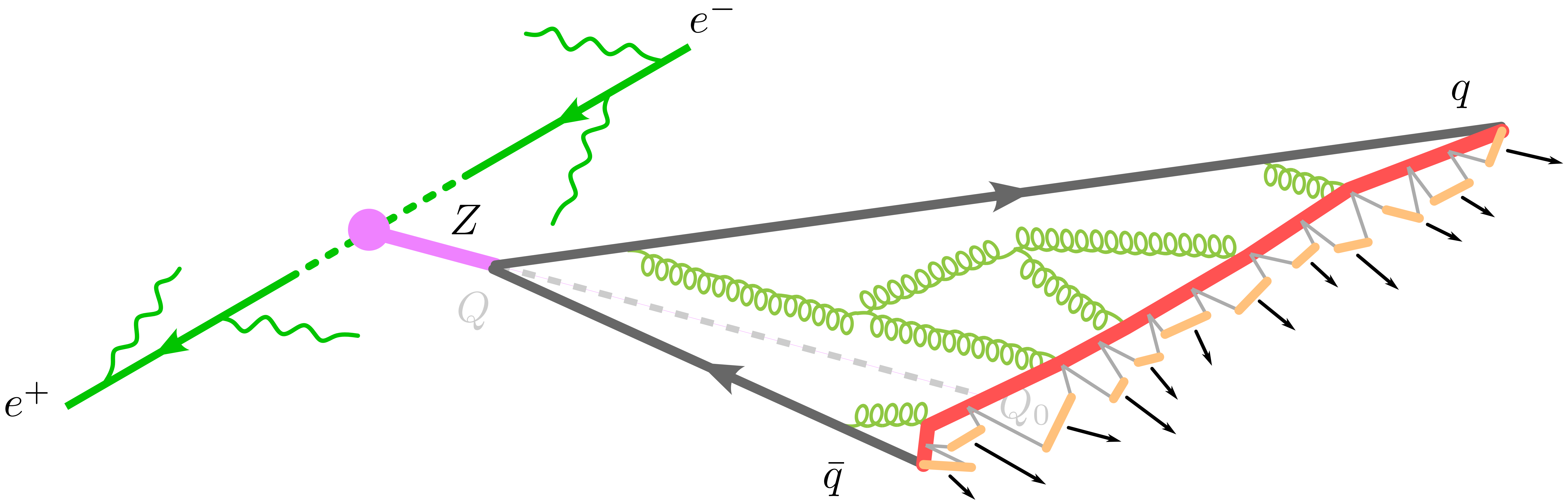
Conservation laws and effective (string-)*h*adronization models

(Based on 2602.12599)



Monash HEP Seminar
March 2nd (3rd), 2026

Tony Menzo, PhD
Joint postdoc @ the University of Alabama and Fermilab

The menu

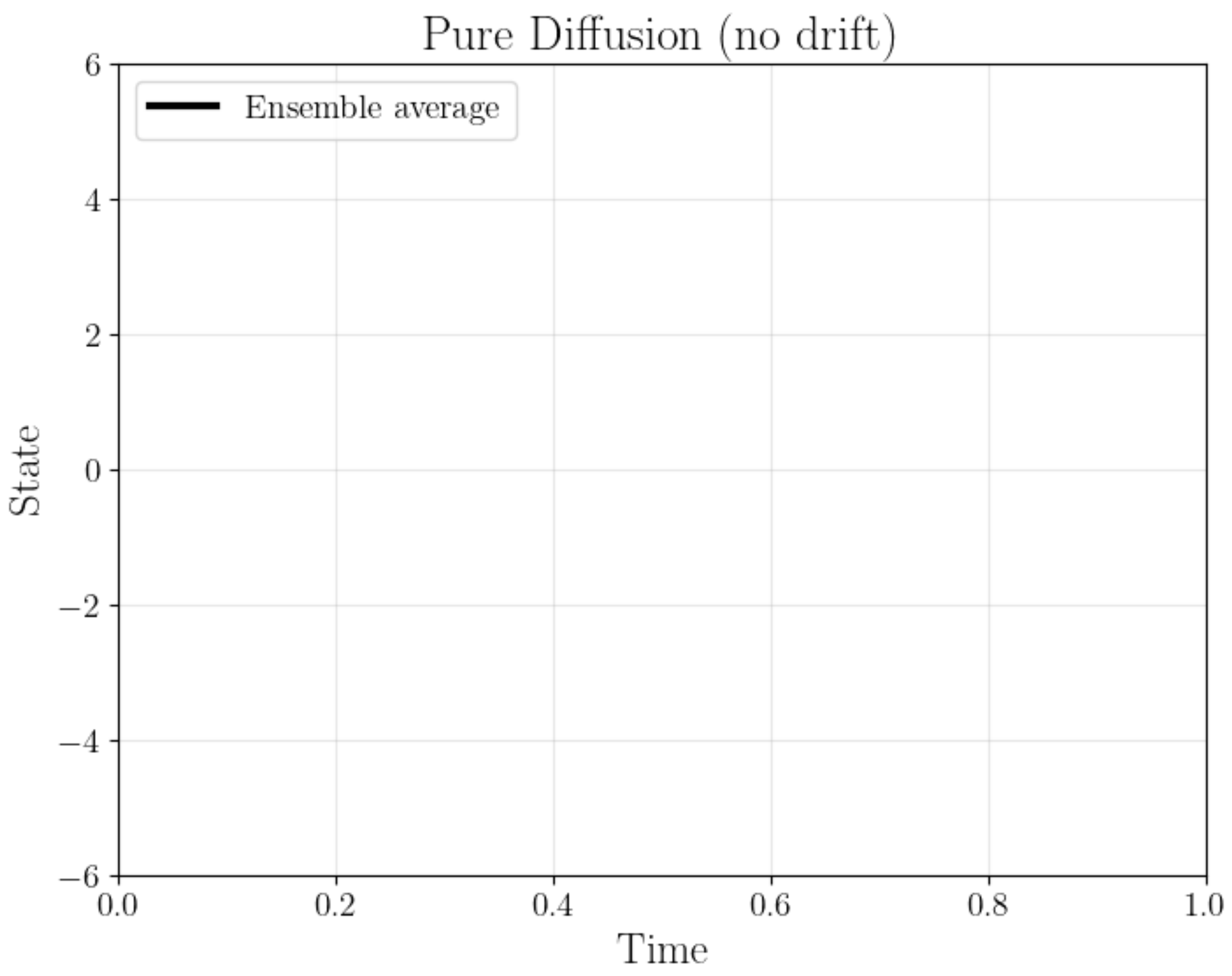
1. Appetizer
   - Stochastic analysis lore
   - Toy model: Brownian motion + constraints
   - Hadronization
2. Three main courses
   - Hadronization as a diffusion process
   - Factorization and effective hadronization models
   - Markovian restoration via Doob h-transform
3. Dessert: Future prospects and conclusions

Stochastic analysis lore

Stochastic analysis is a technology for extracting deterministic statements from stochastic systems.

The system ingredients:

- **State** S – a variable describing the system (e.g. position)
- **Evolution parameter** t – an ordering parameter (e.g. time)
- **Transition kernel** $\mathscr{K}$ – how the state updates, with some stochastic component



Stochastic analysis lore

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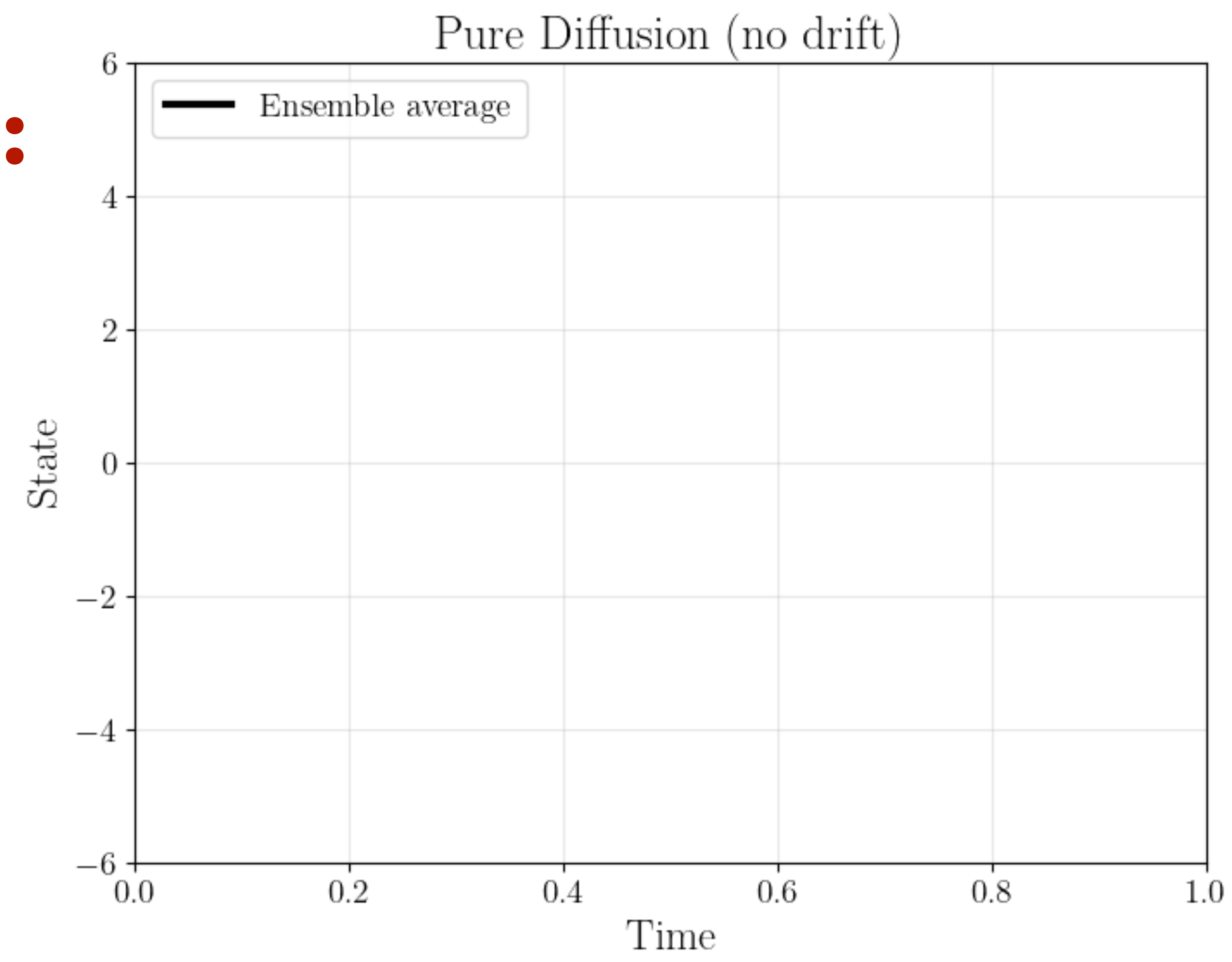
Brownian motion (pure diffusion):

$$S_{n+1} = S_n + \sqrt{\Delta t}\xi_n$$

$$\xi_n \sim \mathcal{N}(0,1)$$

$$\langle S \rangle = 0, \quad \mathrm{Var}(S) = n\Delta t$$



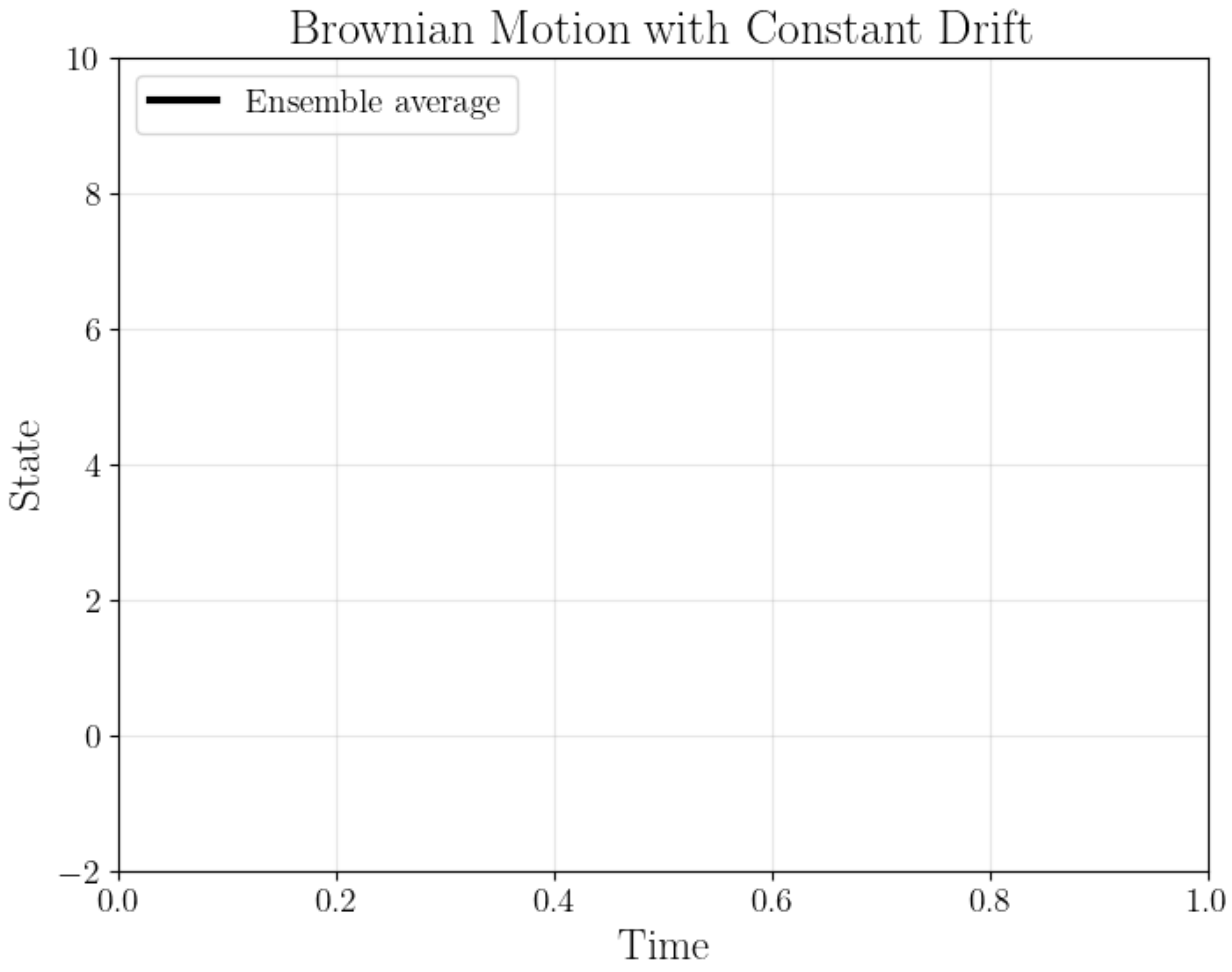
Stochastic analysis lore

Stochastic analysis is a technology for extracting deterministic statements from stochastic systems.

Arithmetic Brownian motion (drift + diffusion):

$$S_{n+1} = S_n + \mu \Delta t + \sigma \sqrt{\Delta t} \xi_n$$

$$\langle S \rangle = S_0 + \mu n \Delta t, \quad \mathrm{Var}(S) = \sigma^2 n \Delta t$$



Stochastic analysis lore

Stochastic analysis is a technology for extracting deterministic statements from stochastic systems.

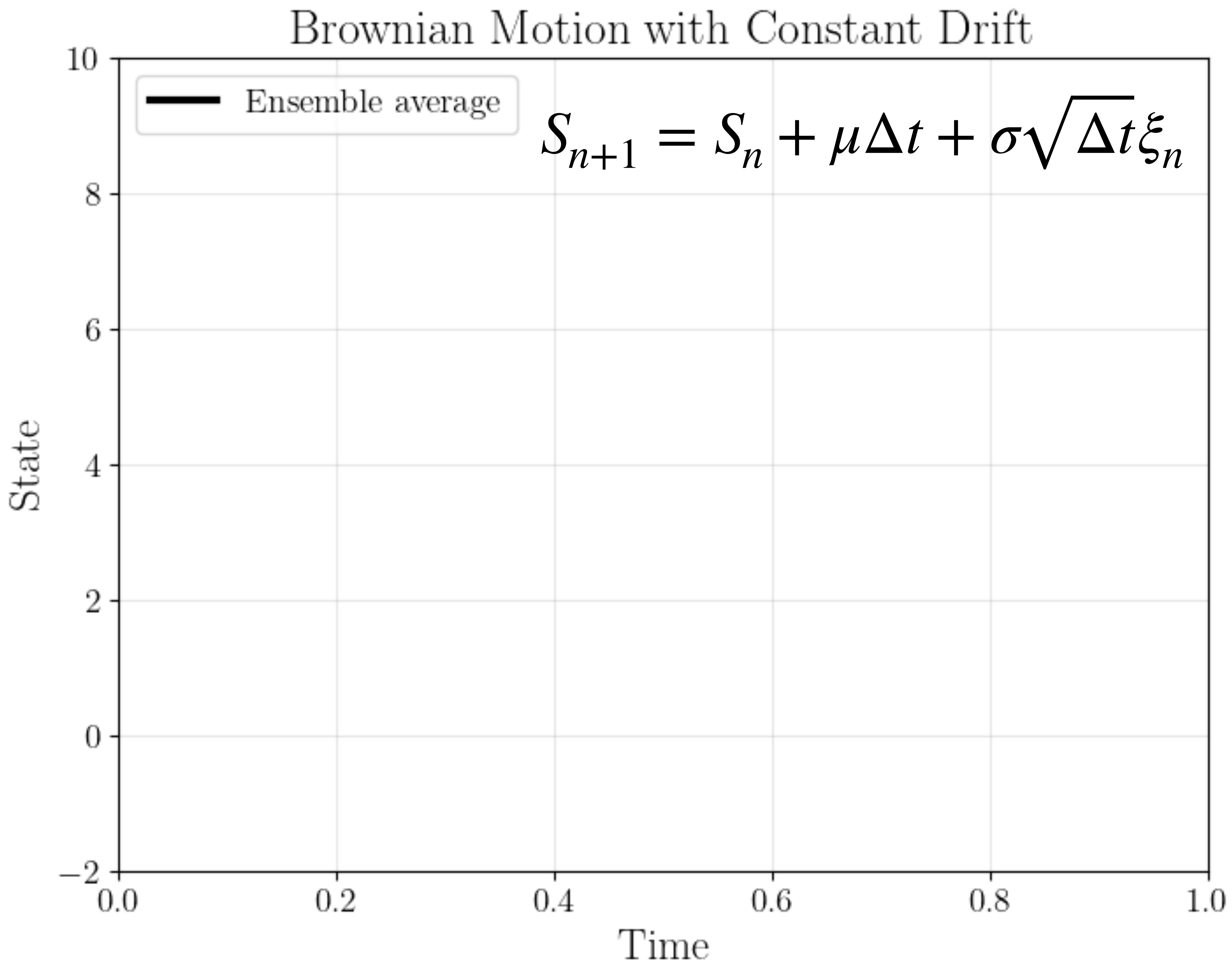
Arithmetic Brownian motion (drift + diffusion):

Consider a single step and its moments:

$$\Delta S \equiv S_{n+1} - S_n$$

$$\langle \Delta S \rangle = \mu \Delta t, \quad \langle (\Delta S)^2 \rangle = \sigma^2 \Delta t$$

$$\boxed{\langle (\Delta S)^k \rangle = c_k (\Delta t)^{k/2}}$$



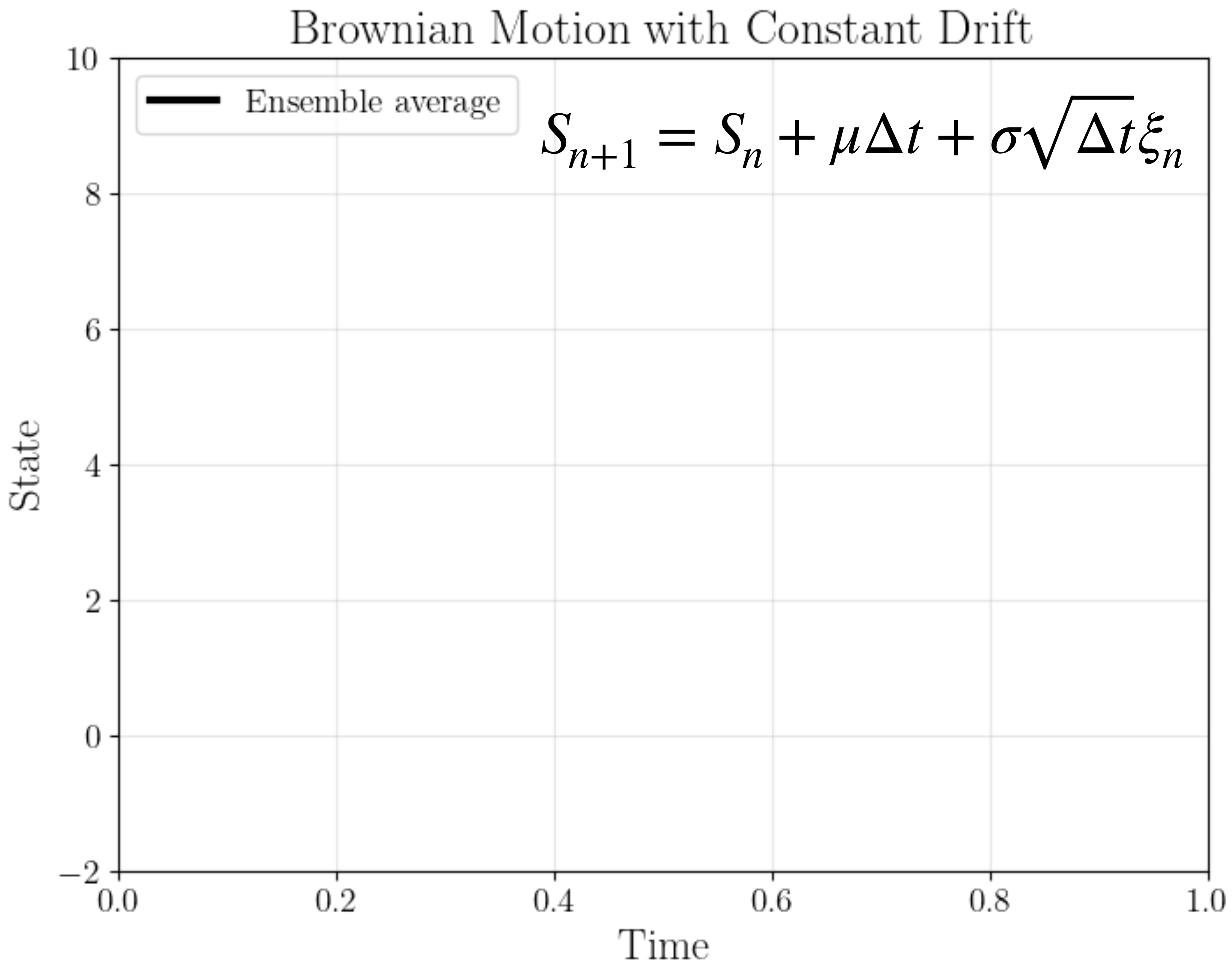
Stochastic analysis lore

Stochastic analysis is a technology for extracting deterministic statements from stochastic systems.

Arithmetic Brownian motion (drift + diffusion):

For $\Delta t \to 0$ the first two moments dominate the dynamics and a *continuum description* emerges

SDE: $$dS_t = \mu dt + \sigma dW_t$$



Stochastic analysis lore

Stochastic analysis is a technology for extracting deterministic statements from stochastic systems.

Arithmetic Brownian motion (drift + diffusion):

What about the evolution of the expectation of a generic observable $\mathcal{O}(S)$?

$$\langle \mathcal{O}(S + dS) \rangle = \langle \mathcal{O}(S) + \mathcal{O}'(S)\, dS + \frac{1}{2}\mathcal{O}''(S)\, dS^2 + \cdots \rangle$$

$$dS = \mu dt + \sigma dW, \quad \mathrm{Var}(dW) = dt$$

$$\frac{d}{dt}\langle \mathcal{O} \rangle = \langle \mu \mathcal{O}' + \frac{1}{2}\sigma^2 \mathcal{O}'' \rangle \equiv \langle \mathcal{L}\mathcal{O} \rangle$$

Stochastic analysis lore

Stochastic analysis is a technology for extracting deterministic statements from stochastic systems.

- **The generator** $\mathscr{L} \equiv \mu \partial_S + \frac{1}{2}\sigma^2 \partial_S^2$ **describes the leading deterministic dynamics**

$$\frac{d}{dt}\langle \mathcal{O}(S) \rangle = \langle \mathscr{L} \mathcal{O} \rangle$$

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- The generator $\mathscr{L} \equiv \mu \partial_S + \frac{1}{2}\sigma^2 \partial_S^2$ describes the leading deterministic dynamics

$$\frac{d}{dt}\langle \mathcal{O}(S) \rangle = \langle \mathscr{L} \mathcal{O} \rangle$$

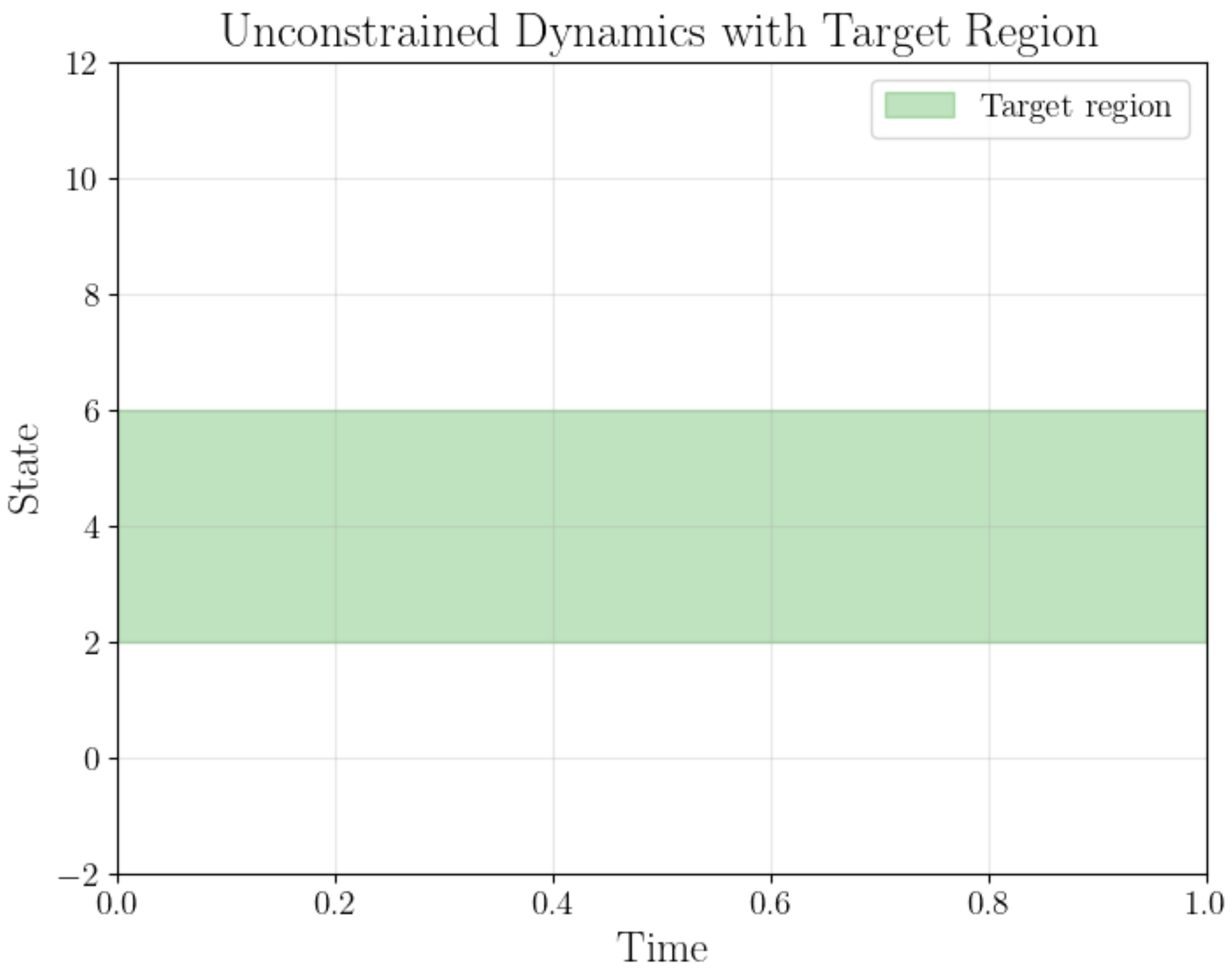
Heisenberg picture in QM:

$$\frac{d}{dt}\langle \mathcal{O} \rangle = \langle i[\mathscr{H}, \mathcal{O}] \rangle$$

The generator $\mathscr{L}$ is the equivalent of the Hamiltonian of the stochastic system!

Constrained brownian motion

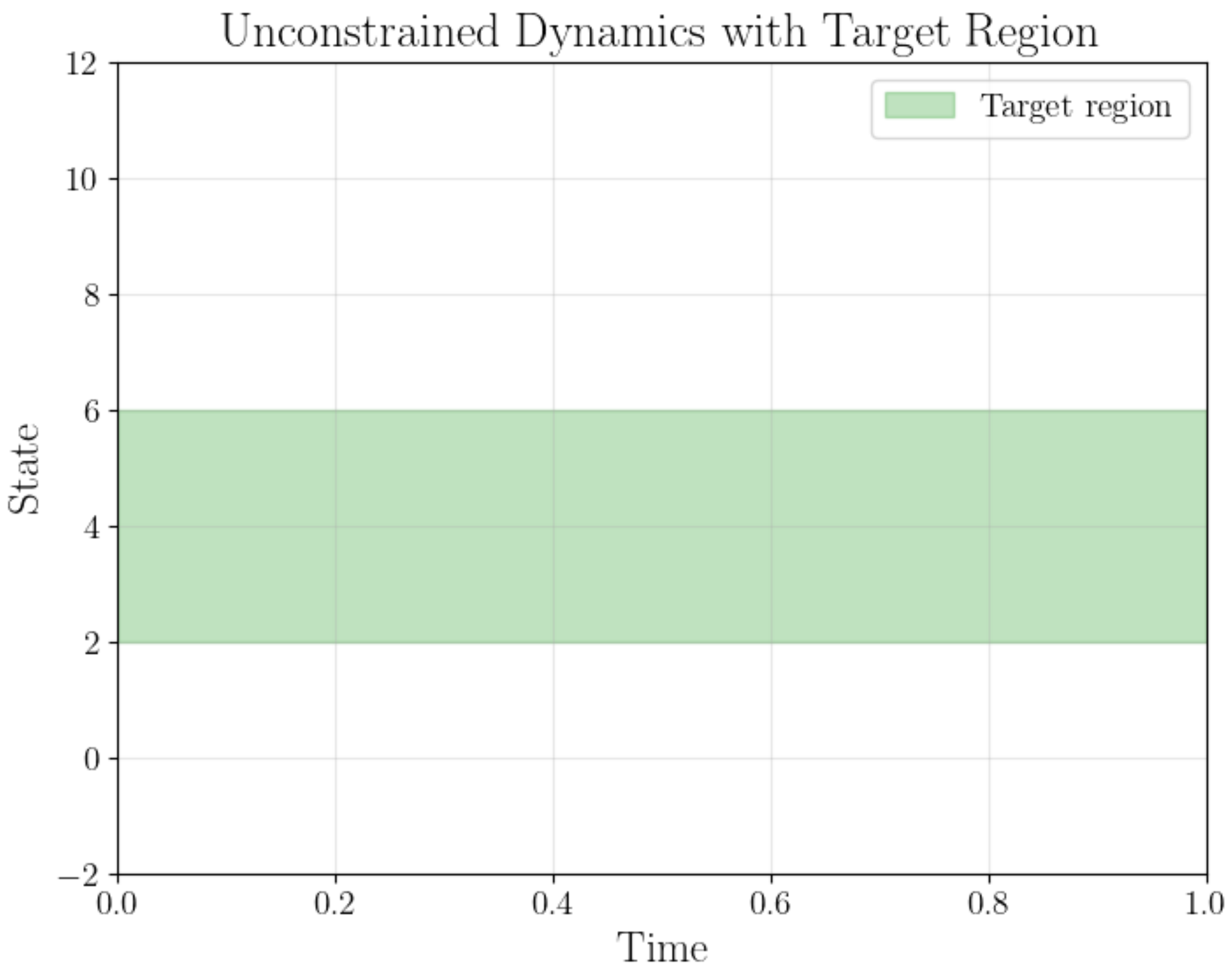
Now consider the same arithmetic Brownian motion but with the additional requirement that trajectories must now land in region $\mathscr{R} \in [\alpha, \beta]$ @ t=T.



Constrained brownian motion

Now consider the same arithmetic Brownian motion but with the additional requirement that trajectories must now land in region $\mathscr{R} \in [\alpha, \beta]$ @ t=T.

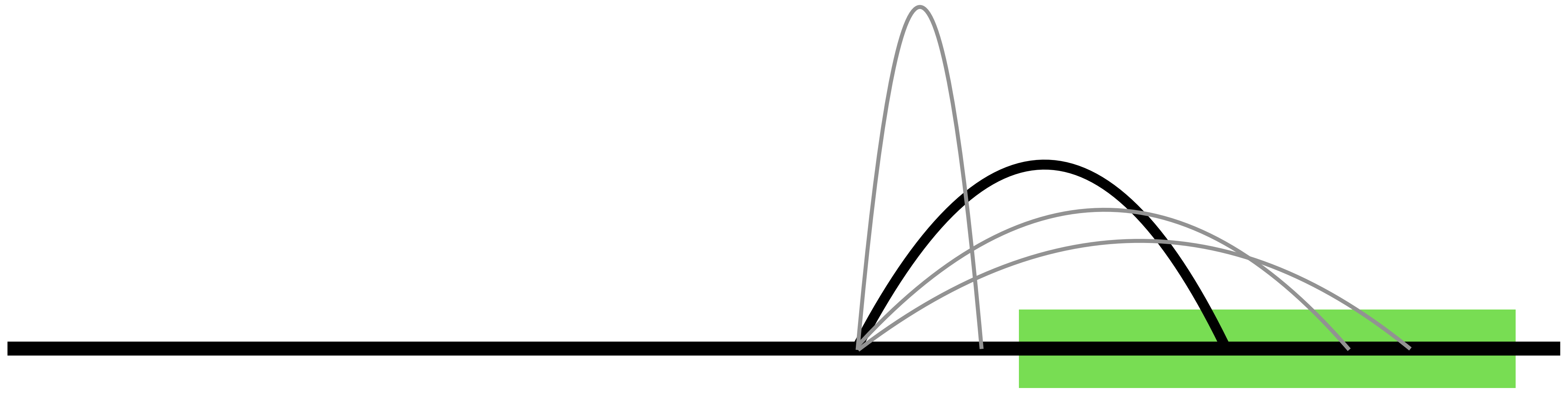
A natural question: given that I'm at an arbitrary state (S,t), what is the probability of reaching the target $\mathscr{R}$?



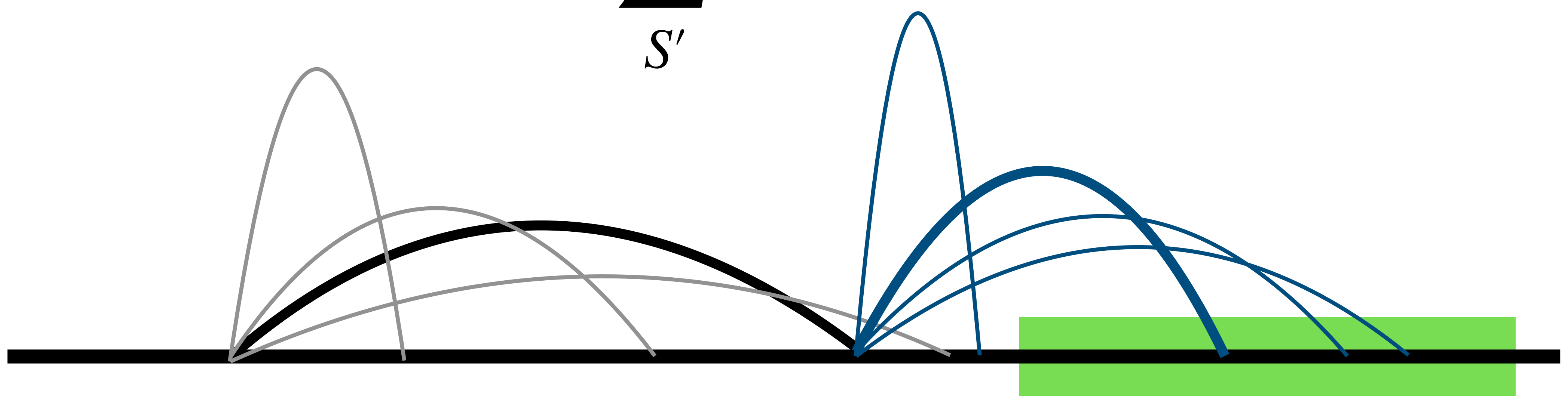
Constrained brownian motion

The success probability

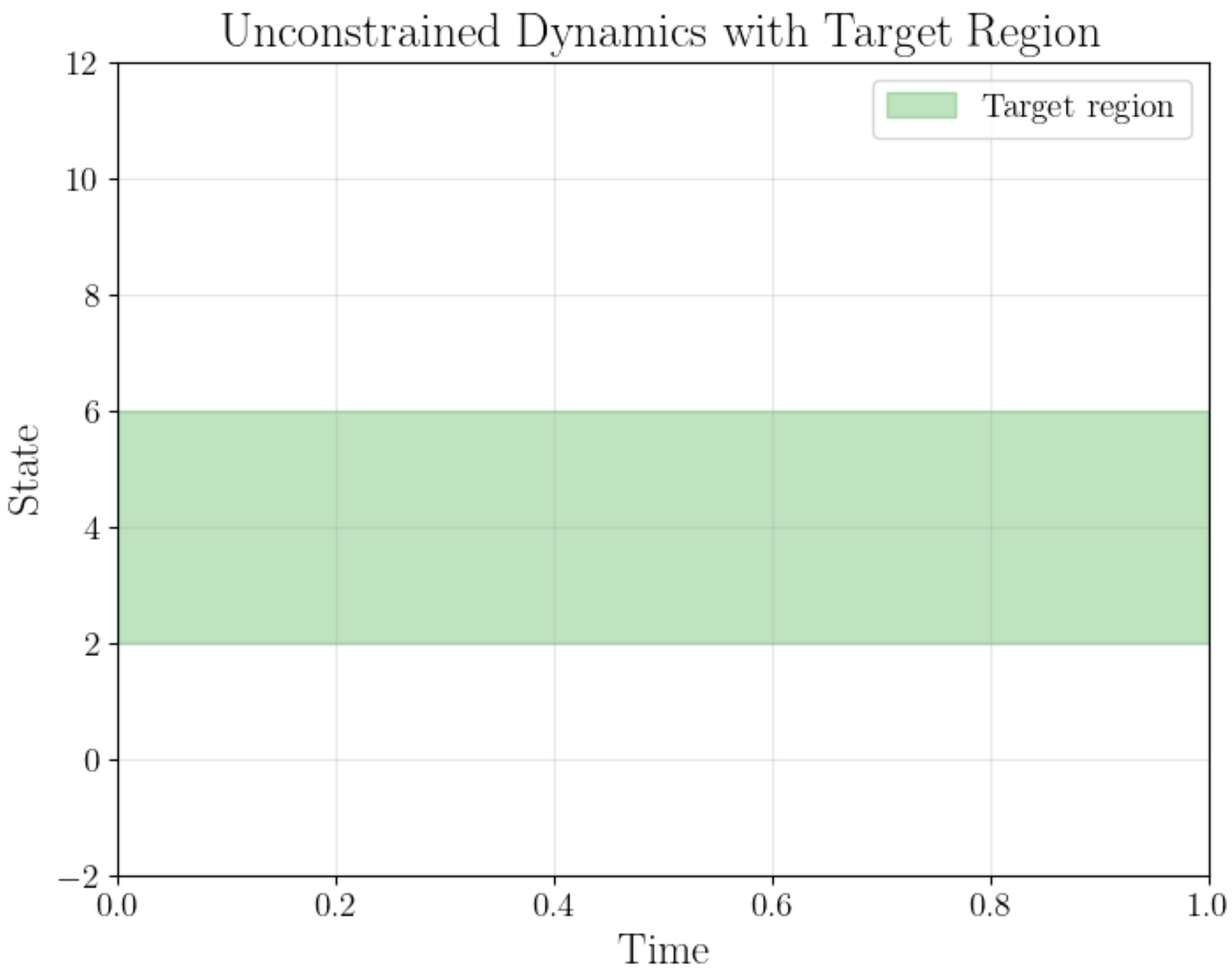
$$h(S,t) = \mathrm{Prob}(\mathrm{success}\,|\,S,t)$$



$$h_{N-1}(S) = \sum_{S'} K(S \to S') h_N(S')$$



$$h_{N-2}(S) = \sum_{S'} K(S \to S') h_{N-1}(S')$$



Constrained brownian motion

The success probability

$$h(S,t) = \text{Prob}(\text{success}\,|\,S,t)$$

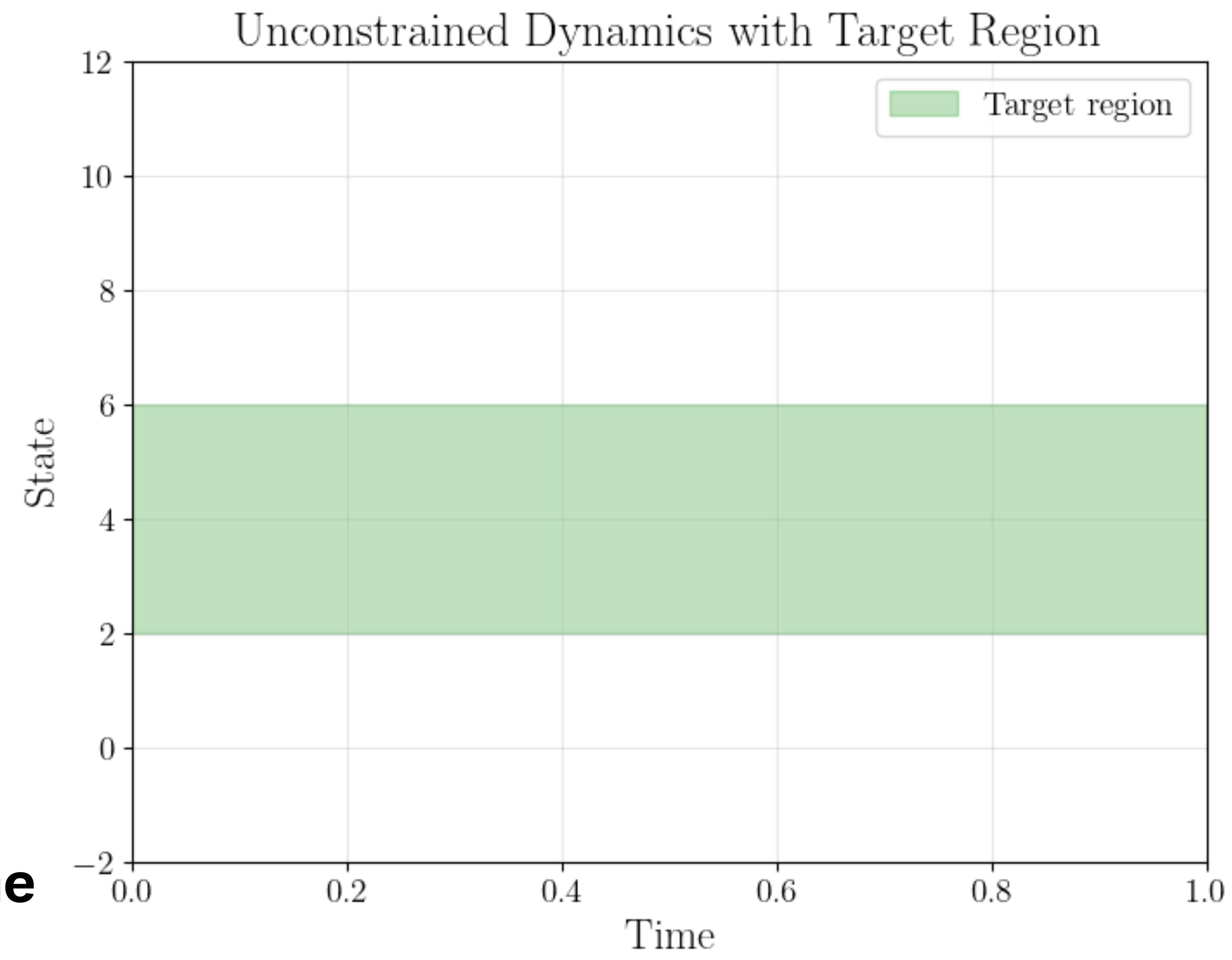
$$0 = \sum_{S'} K(S \to S')[h_{n+1}(S') - h_n(S)] \equiv \langle \Delta h \rangle$$

$$\langle \Delta h \rangle = 0$$

$$\downarrow$$

$$(\partial_t + \mathscr{L})\, h = 0$$

h **is known as a *martingale observable*. It is invariant under the stochastic dynamics.**



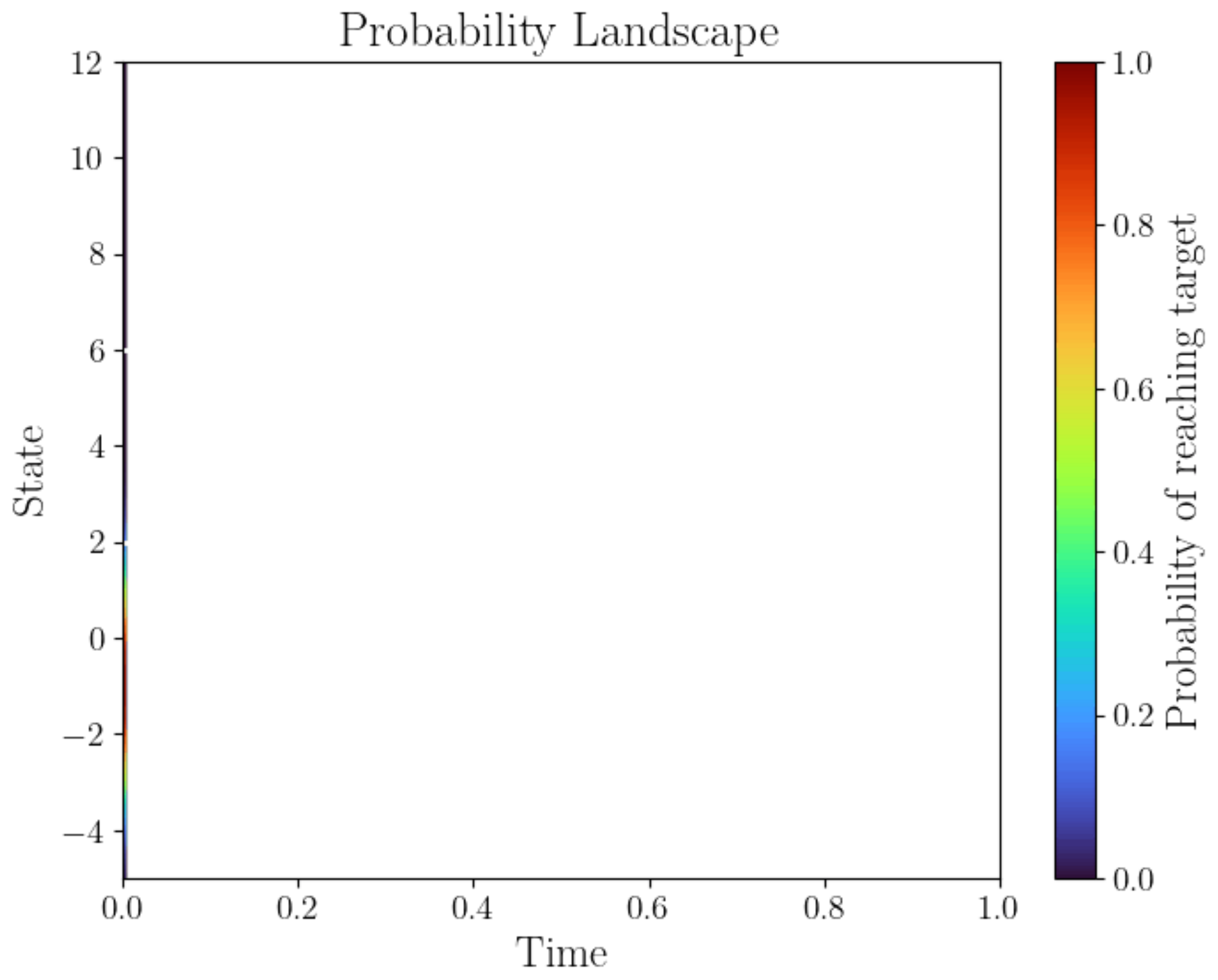
Constrained brownian motion

Taking the diffusion limit

$$\partial_t h + \mu h' + \frac{1}{2}\sigma^2 h'' = 0$$

↓

$$h(S,t) = \Phi\left(\frac{\beta - S - b(T-t)}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{\alpha - S - b(T-t)}{\sigma\sqrt{T-t}}\right)$$



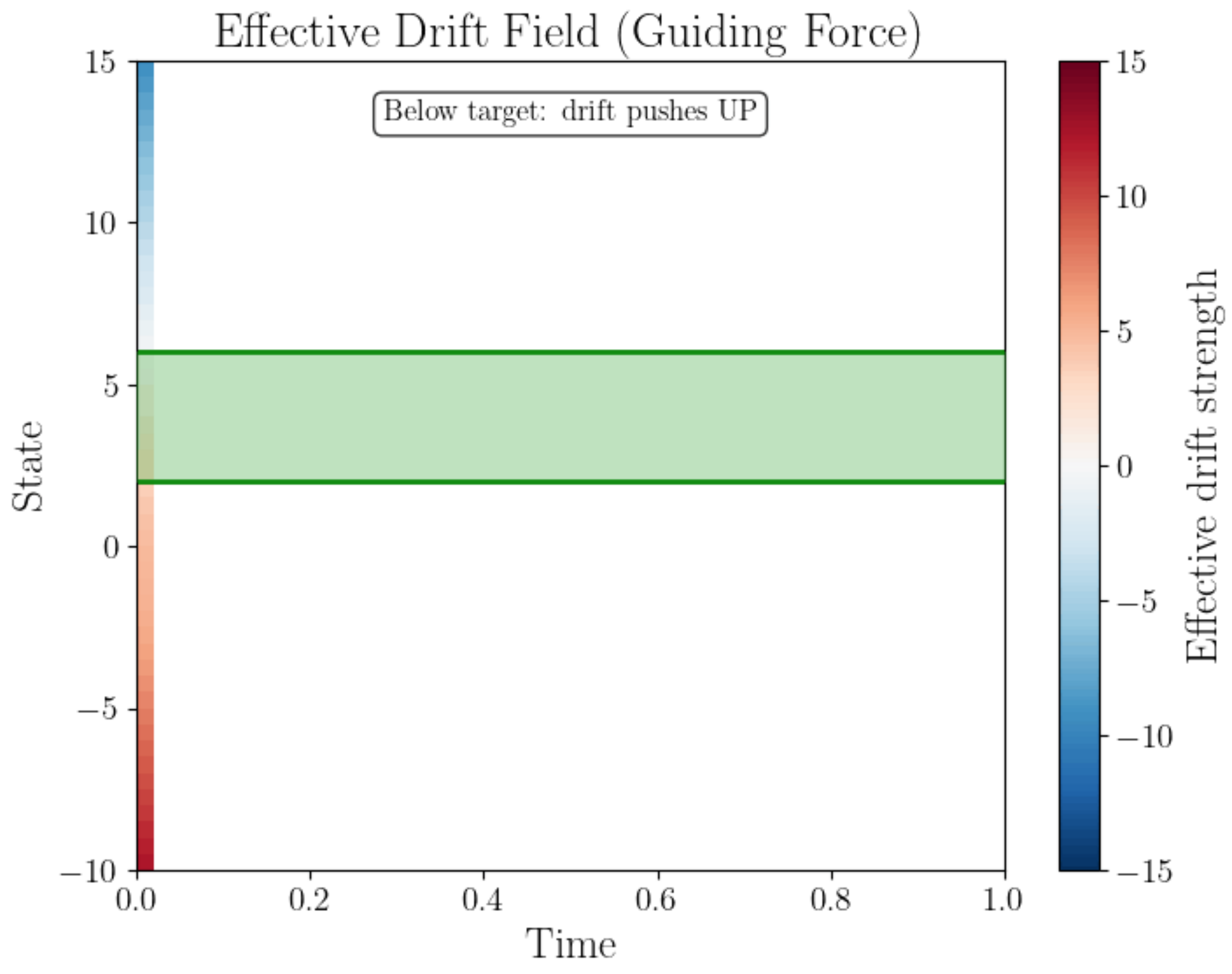
Constrained brownian motion

In QM, observables which commute with the Hamiltonian $[H, Q] = 0$, indicate a conserved charge and in many cases indicate a clean basis to work in. Martingale observables do the same for stochastic dynamics. We can perform a similarity transform on our generator

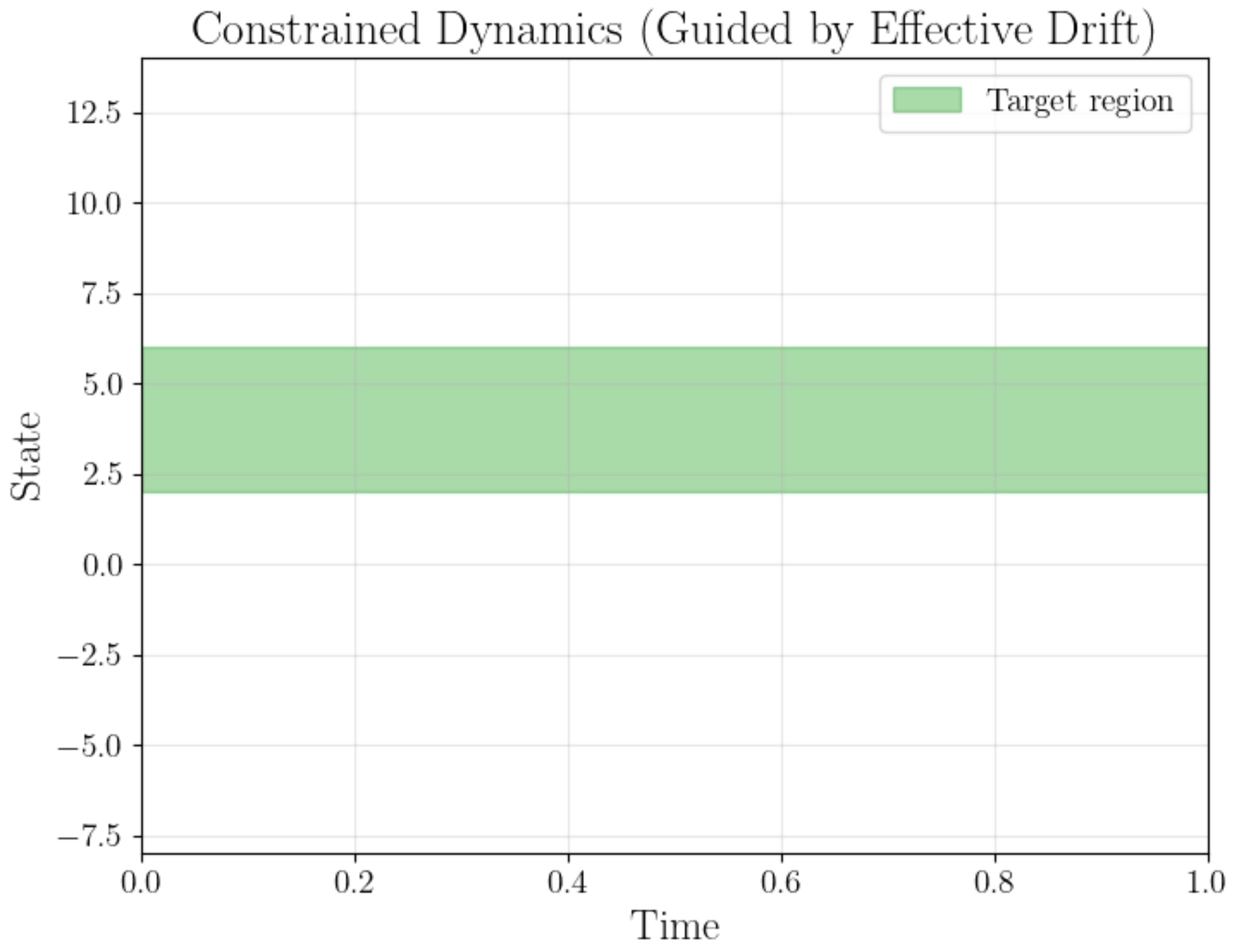
$$\mathscr{L}_h = h^{-1}\mathscr{L}h$$

an effect which can be absorbed into a renormalization of the bare drift

$$b \to b + \sigma^2 \partial_S \ln h$$



Constrained brownian motion



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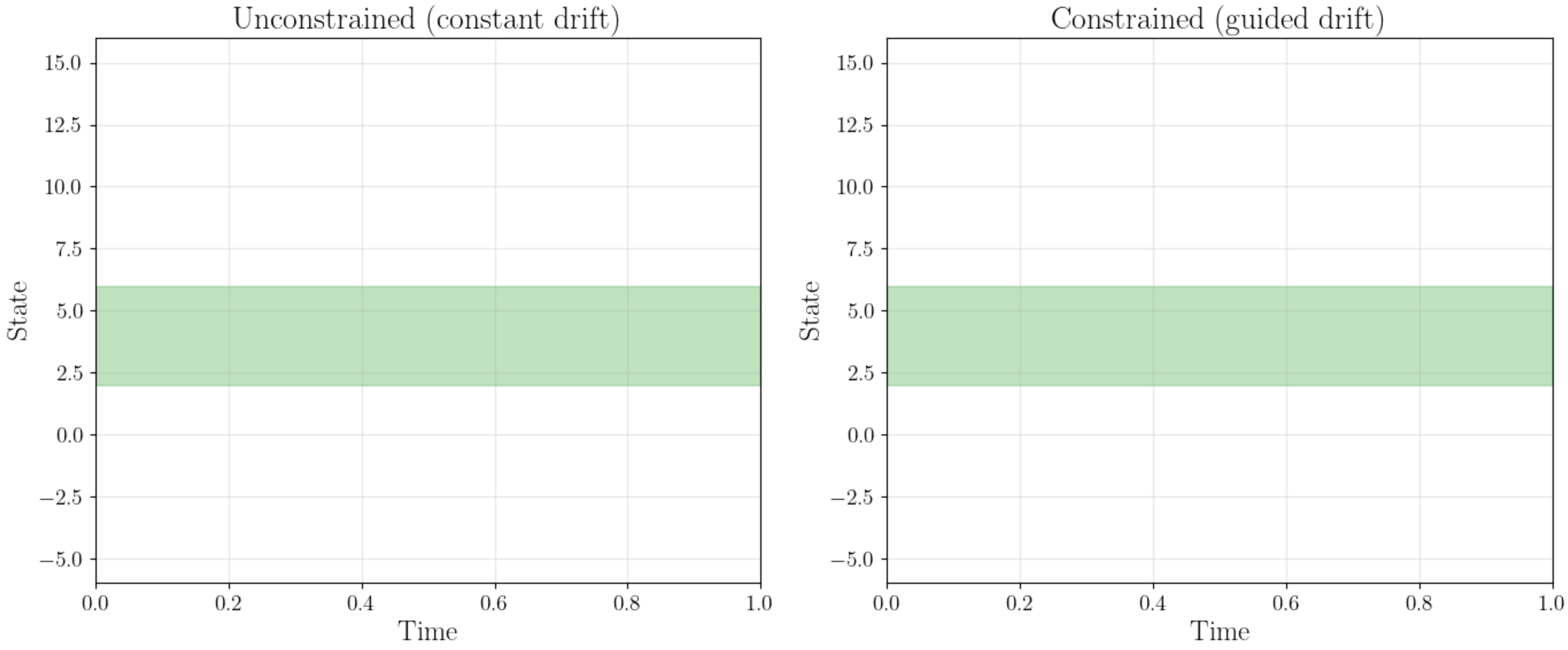
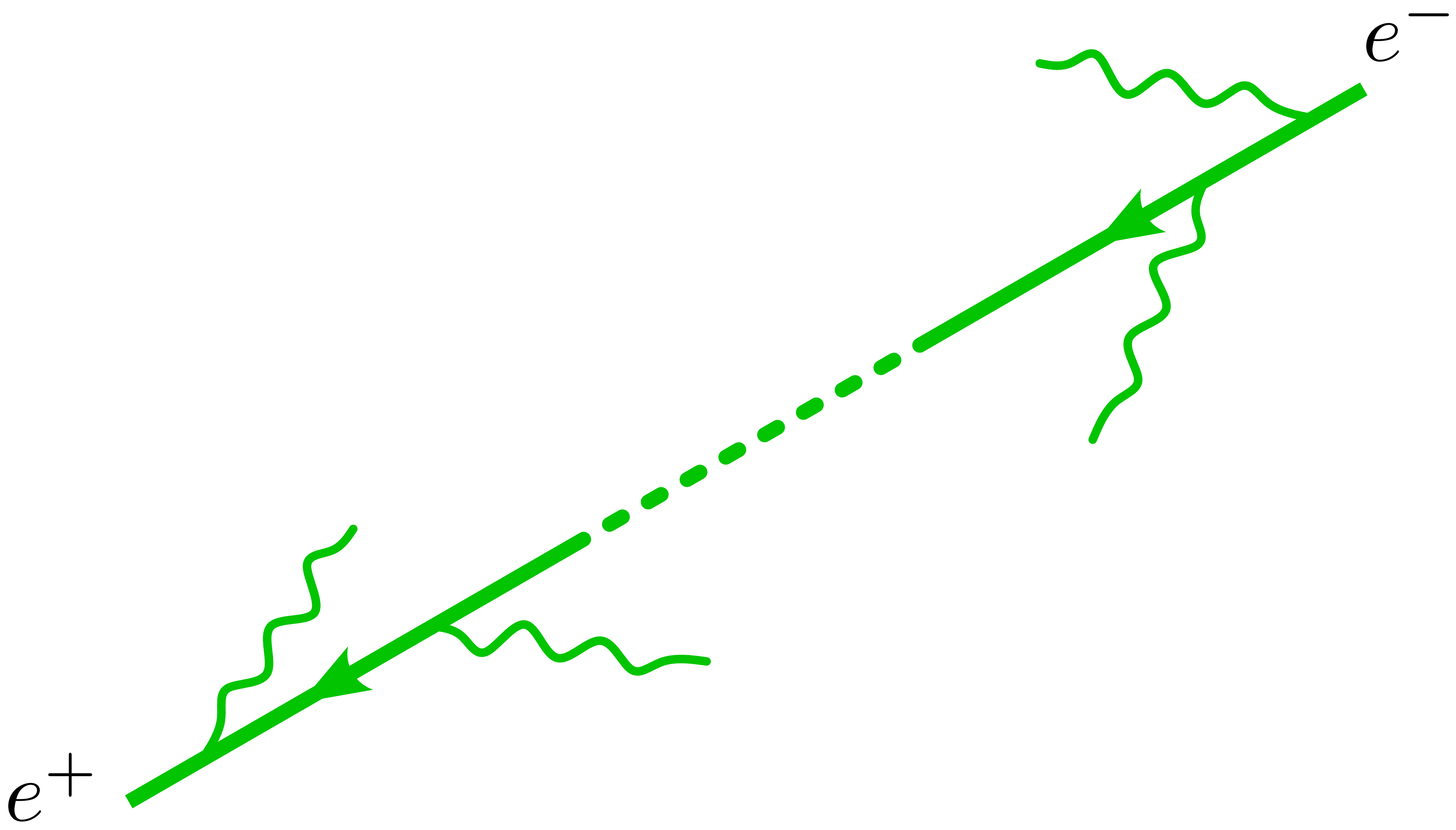


Hadronization

Converting colored partons to color-singlet hadrons



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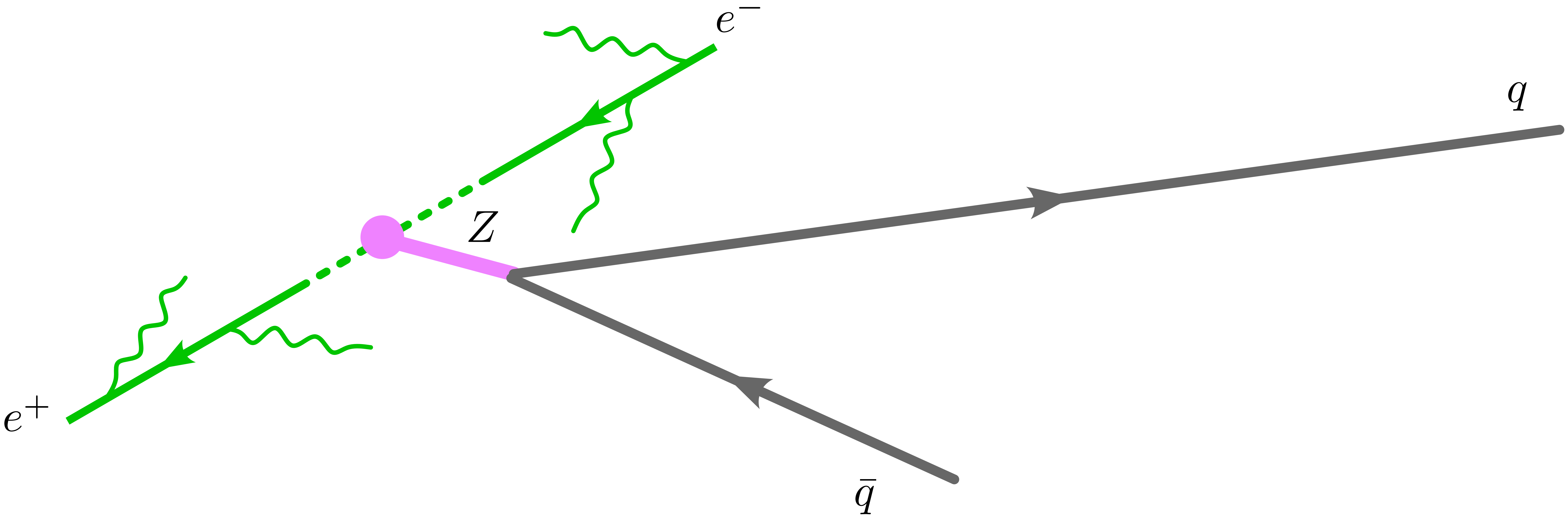

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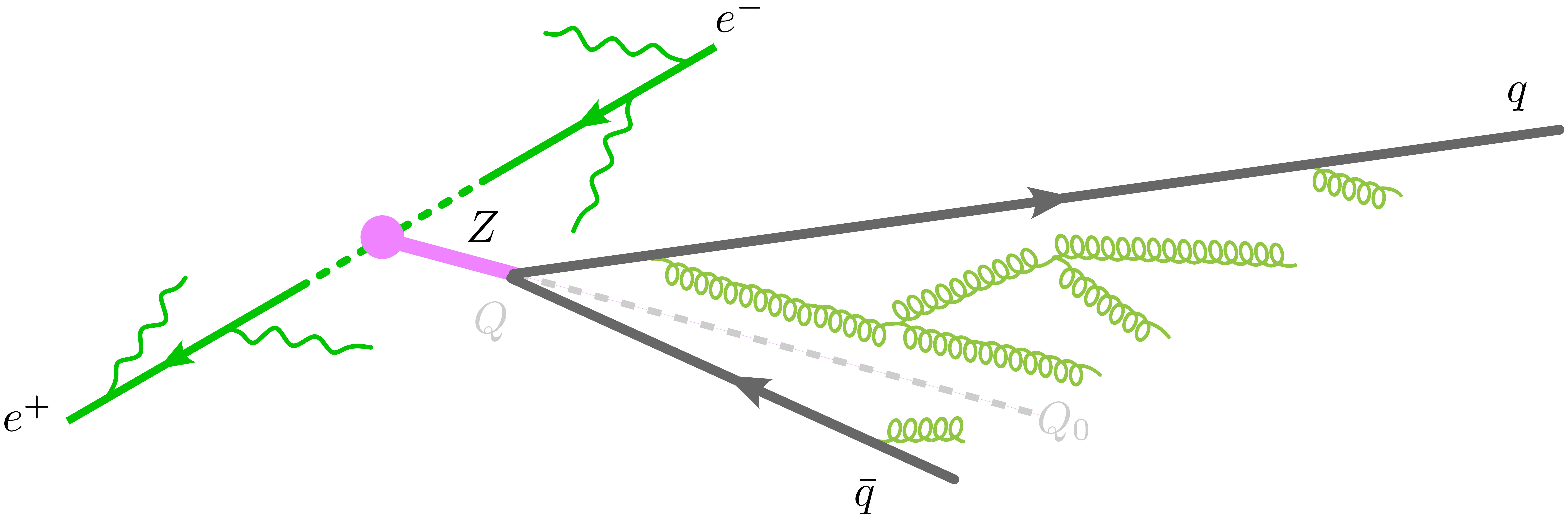

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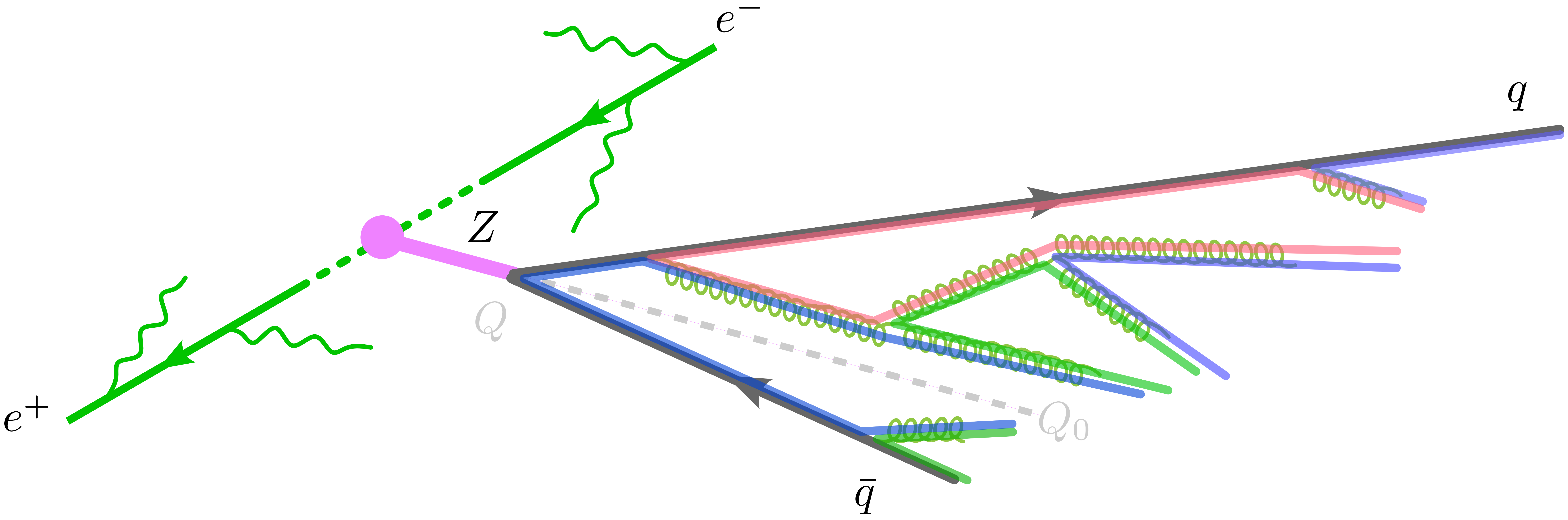

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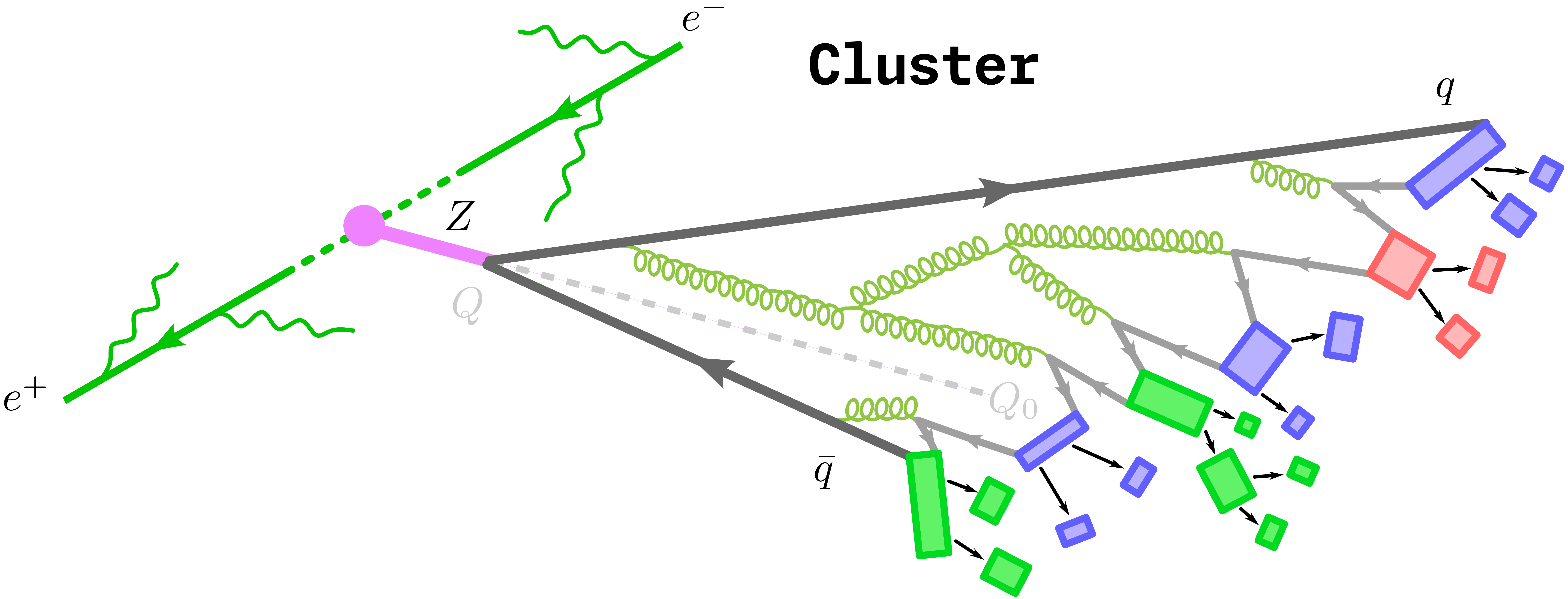

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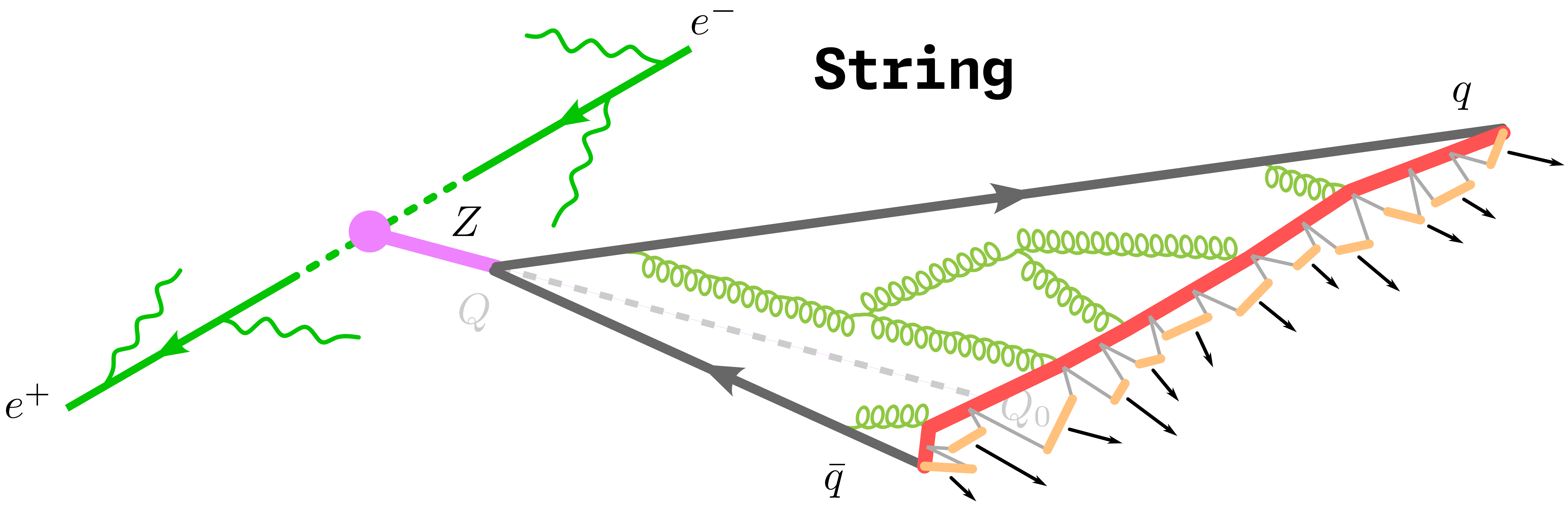
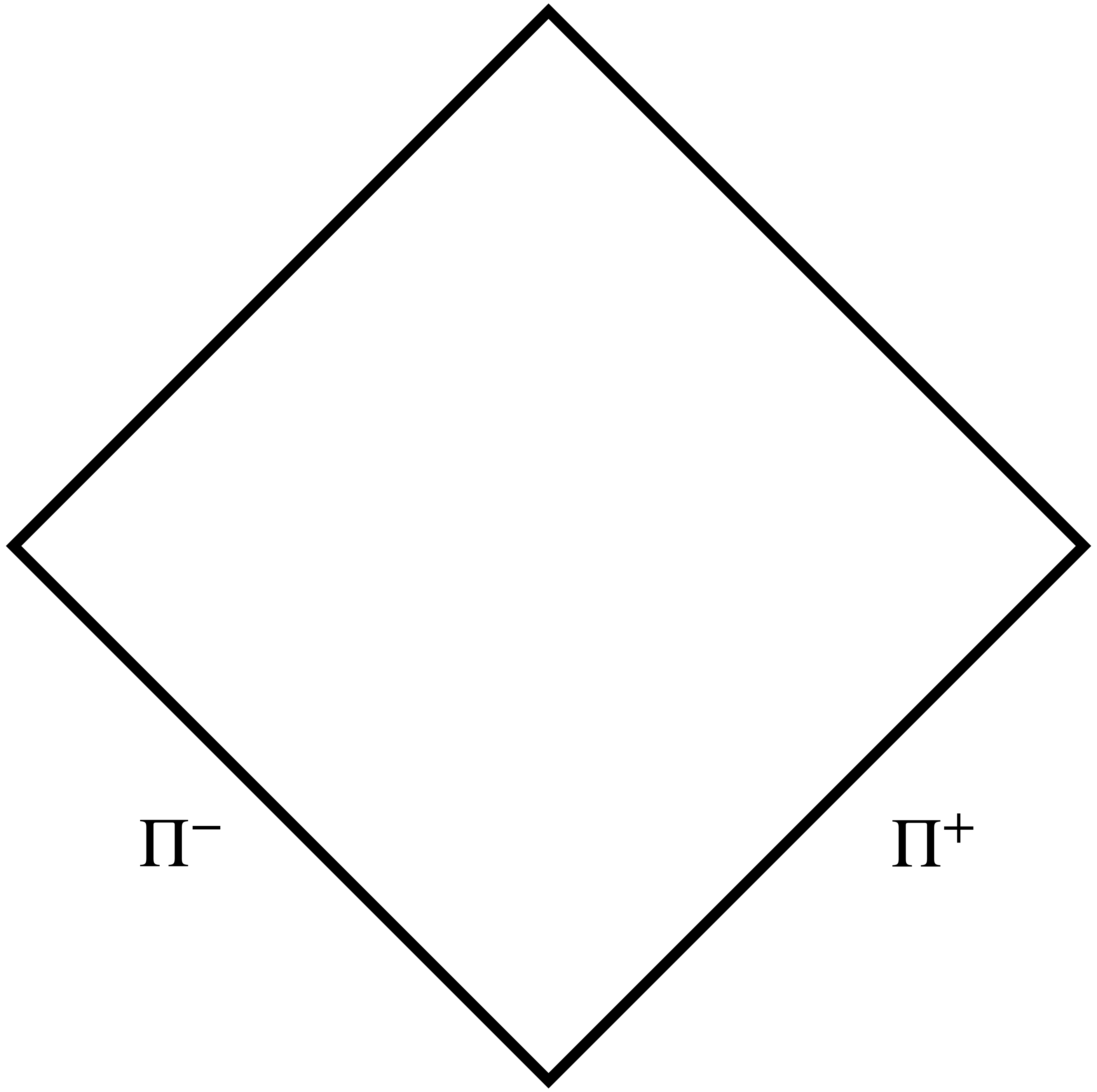


String hadronization

Massless quarks, pion-philic hadronization in the chiral limit ($m_\pi = 0$)

Consider a $q\bar{q}$ system aligned along the z-axis. The string is characterized by its light-cone momenta $\Pi^{\pm} \equiv E \pm p_z$, and invariant mass $M^2 = \Pi^+\Pi^-$.



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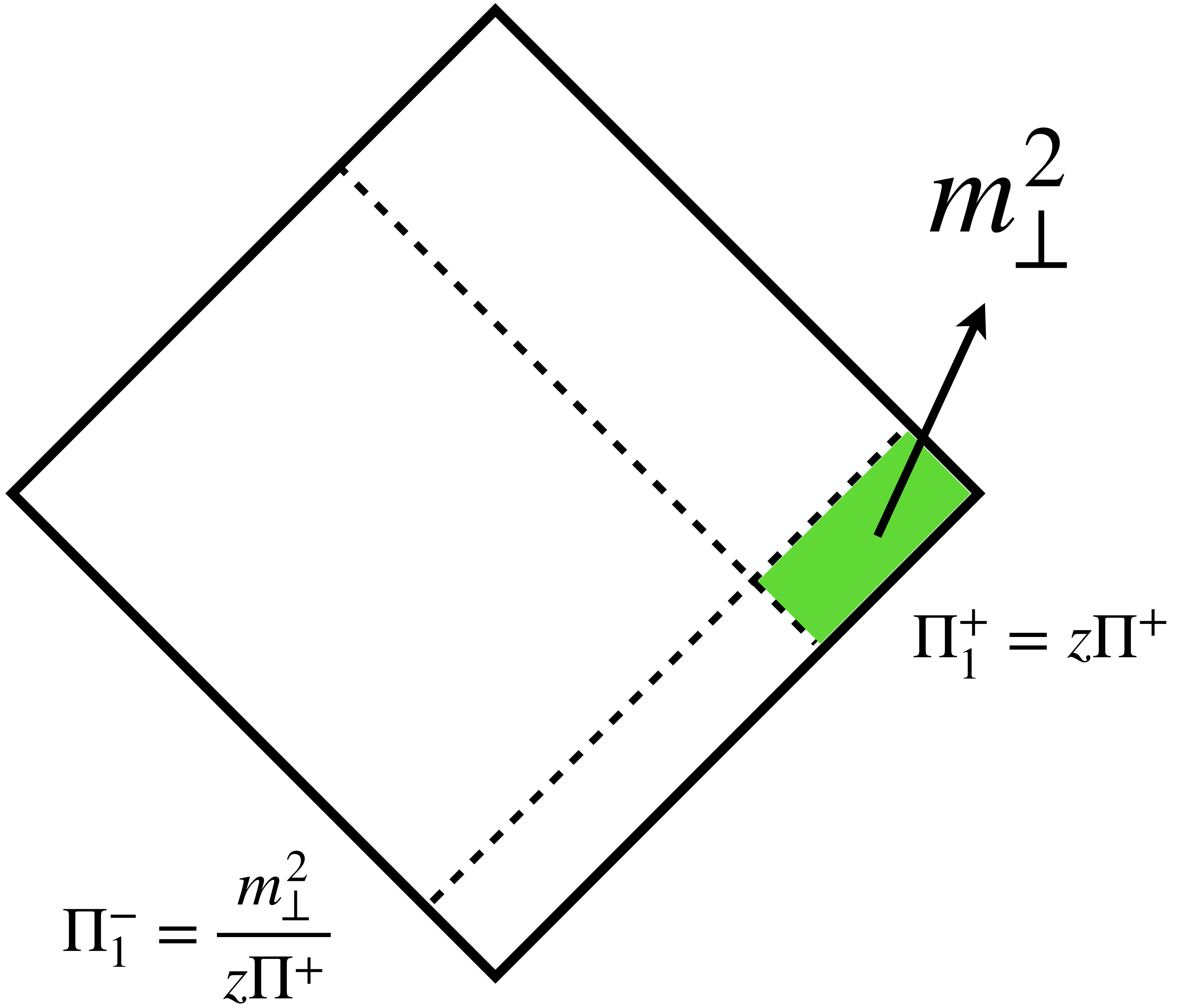
One iteration: sample flavor and p_T of the string break (giving $m_\perp^2$) as well as the longitudinal light-cone momentum fraction z.

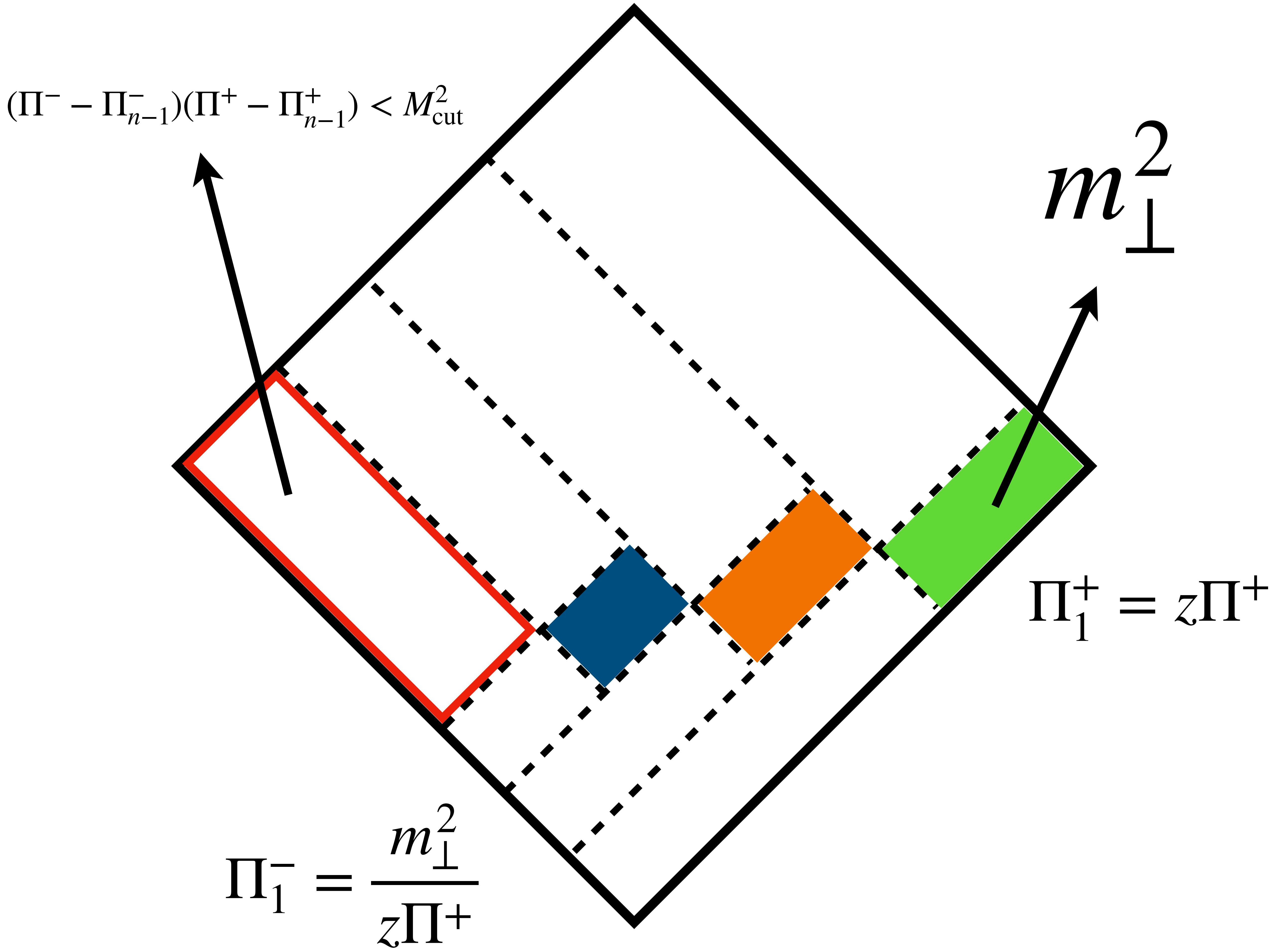


String hadronization

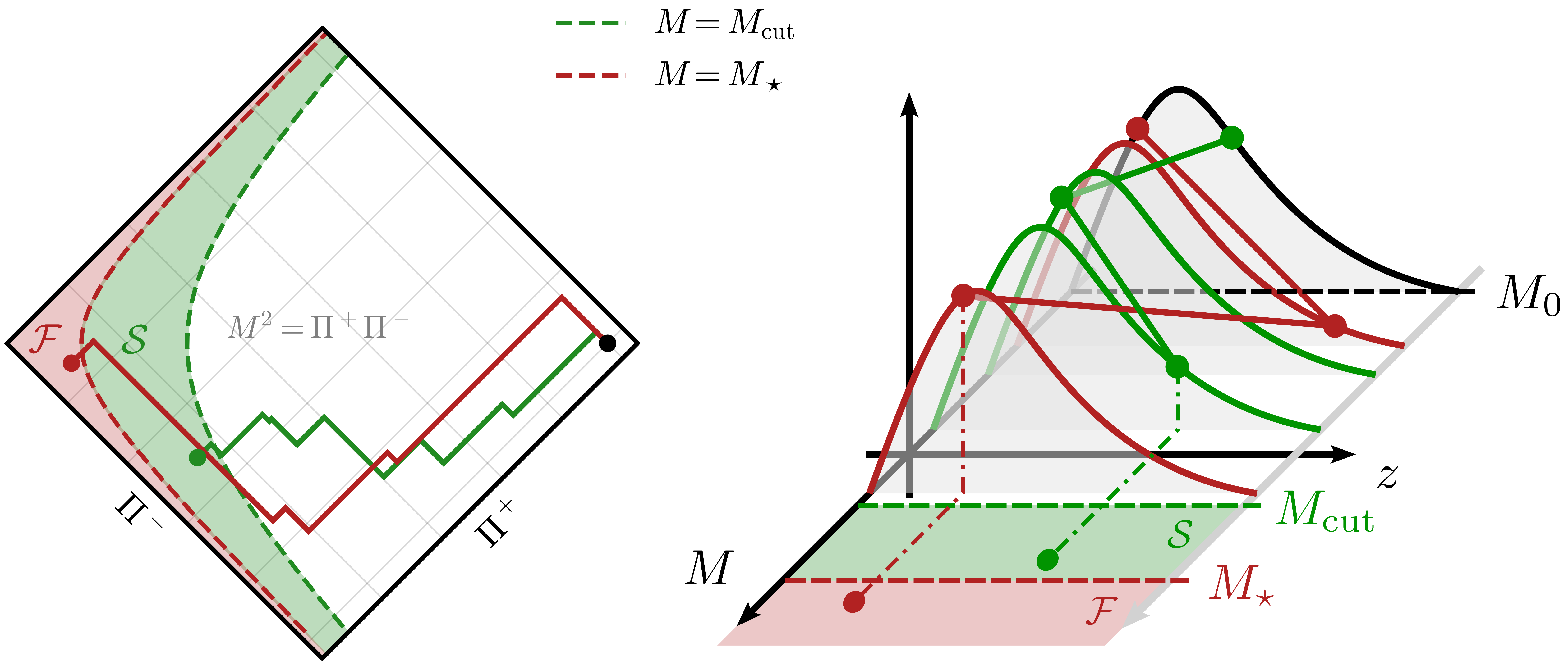
Massless quarks, pion-philic hadronization in the chiral limit ($m_\pi = 0$)

One iteration: sample flavor and p_T of the string break (giving $m_\perp^2$) as well as the longitudinal light-cone momentum fraction z.

Iterate until the remaining string mass drops below some threshold $M_{\rm cut}$, then call some cluster decay



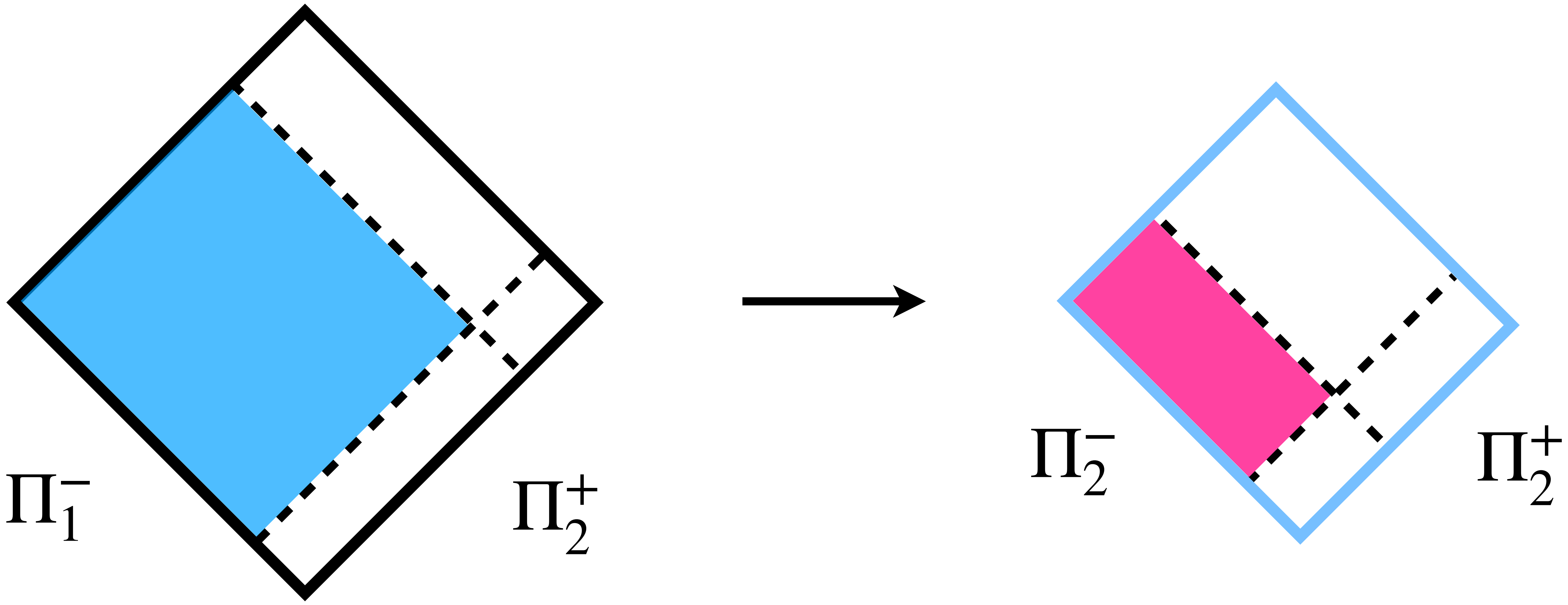
Hadronization as a stochastic process



Hadronization as a stochastic process

Consider "invariant mass" $M^2 = \Pi^+\Pi^-$ as the state variable

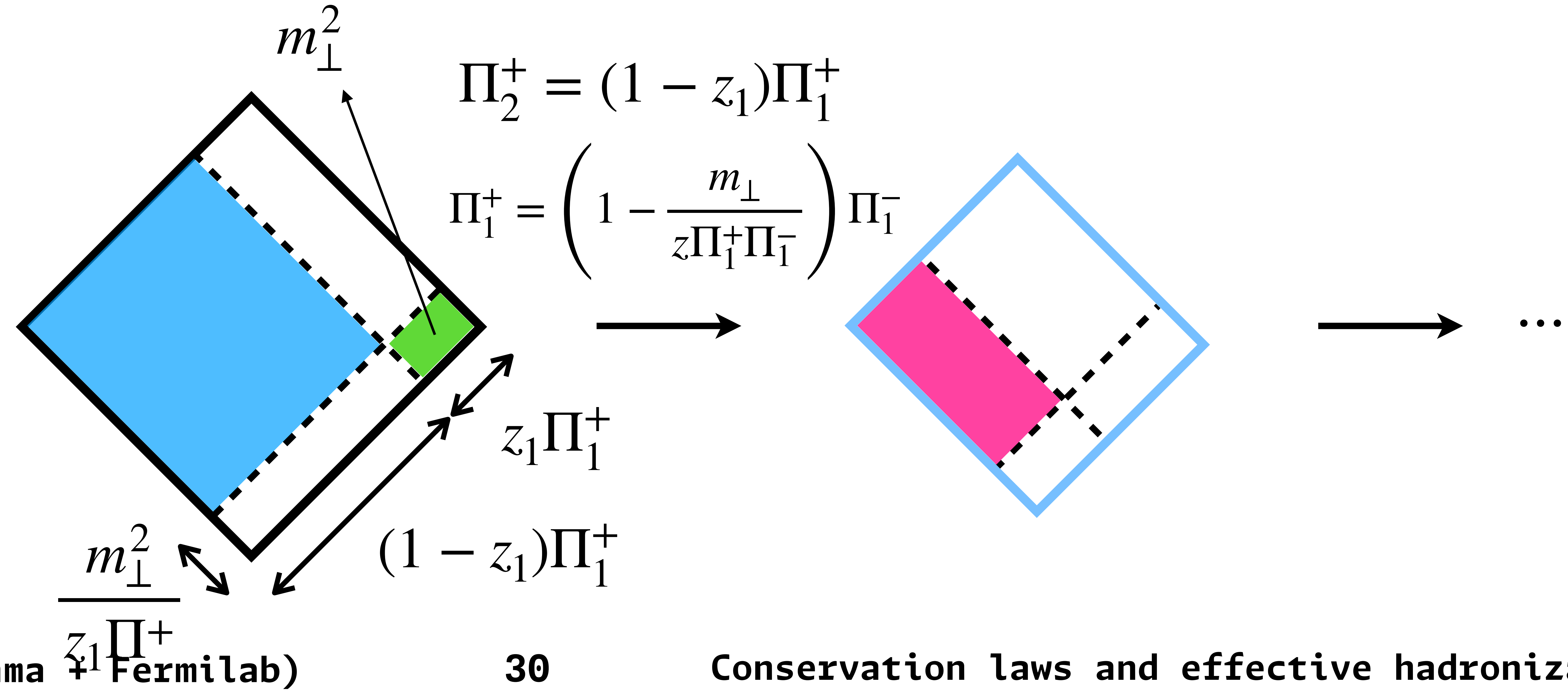
$$M'^2 = \left(M^2 - \frac{m_\perp^2}{z}\right)(1-z) \quad - p_T^2$$



Hadronization as a stochastic process

Sample z according to ***joint*** distribution

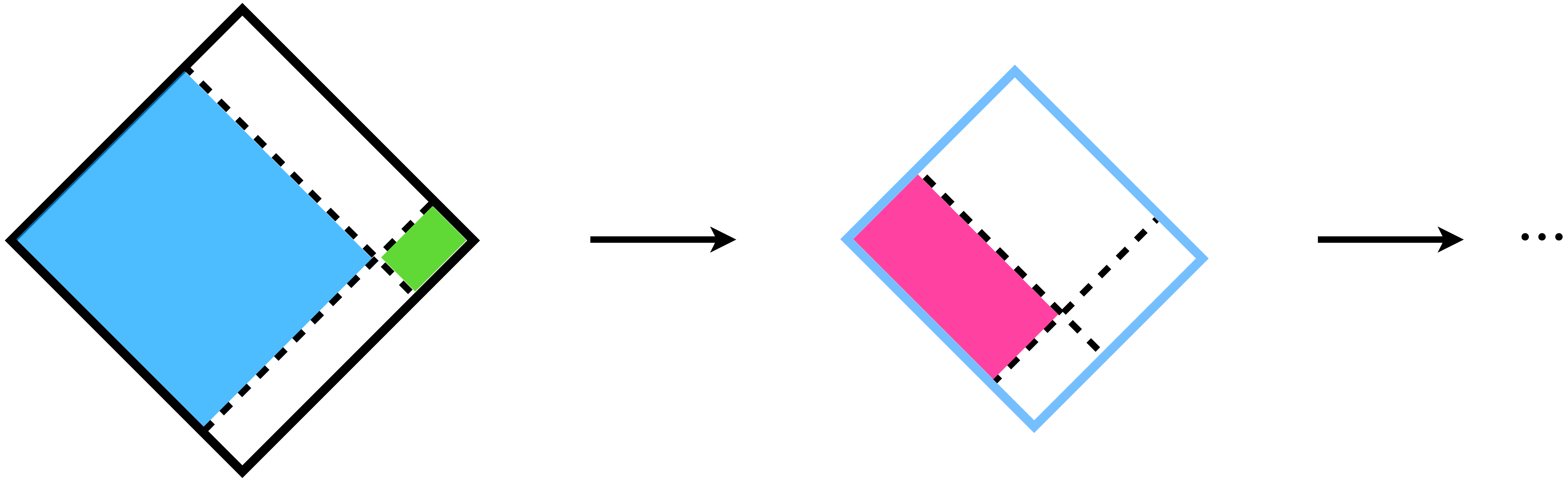
$$f(z, m_\perp^2) \propto \frac{(1-z)^a}{z} e^{\frac{-bm_\perp^2}{z}}$$



Hadronization as a stochastic process

The *kernel* encodes all possible ways of reaching M' from M via $f(z, m_\perp^2)$

$$\mathscr{K}(M \to M') = \int dz dm_\perp^2 f(z, m_\perp^2) \Theta\left(M'^2 - (M^2 - m_\perp^2/z)(1-z)\right)$$



Hadronization as a stochastic process

In the limit where fragmentations occur “many” times, the dynamics admit a continuum description generated by $\mathscr{L}$

$$\mathscr{L} \equiv \mu \partial_M + \frac{1}{2}\sigma^2 \partial_M^2 + \cdots$$

where all dynamics are governed by the transport coefficients

$$\mu \equiv \langle \Delta M \rangle, \quad \sigma^2 = \langle (\Delta M)^2 \rangle - \mu^2$$

$$\Delta M = M' - M = M \left(\sqrt{1 - z - \frac{m_\perp^2}{M^2}\left(\frac{1-z}{z}\right)} - 1 \right)$$

Factorization

The game: compute moments of $\Delta M = M' - M$

Define

$$u \equiv z, \quad v \equiv \frac{m_\perp^2}{zM^2}, \quad v \in \left[\frac{m_\star}{uM^2}, 1\right]$$

such that

$$M' = M\sqrt{(1-u)(1-v)}\,.$$

In the chiral limit $m_\star \sim m_\pi \to 0$

$$f(z\,|\,M) = p(u)p(v\,|\,M)$$

with

$$p(u) = (1+a)(1-u)^a, \quad p(v\,|\,M) = \frac{bM^2 e^{-bM^2 v}}{1 - e^{-bM^2}}$$

Factorization

The game: compute moments of $\Delta M = M' - M = M\left(\sqrt{1 - z - \frac{m_\perp^2}{M^2}\left(\frac{1-z}{z}\right)} - 1\right)$

With factorization the μ and σ^2 calculation is very clean

$$M' = M\sqrt{(1-u)}\sqrt{(1-v)}$$

$$\frac{\mu(M)}{M} = -\frac{1}{2}\big(\langle u\rangle + \langle v\rangle\big) + \frac{1}{4}\langle uv\rangle - \frac{1}{8}\big(\langle u^2\rangle + \langle v^2\rangle\big) + \cdots,$$

$$\frac{\sigma^2(M)}{M^2} = \frac{1}{4}\big(\langle u^2\rangle + \langle v^2\rangle + 2\langle uv\rangle\big) - \frac{1}{4}\big(\langle u\rangle + \langle v\rangle\big)^2$$

with $\langle u^q v^p\rangle = \langle u^q\rangle_u \langle v^p\rangle_v$.

Factorization

Arbitrary moments of u and v can be computed exactly

$$\langle u^p \rangle = \mathbb{E}[u^p] = \frac{p!}{\prod_{k=2}^{p+1}(a+k)}, \quad \langle v^q \rangle = \mathbb{E}[v^q] = \frac{q!}{\lambda^q(1-e^{-\lambda})}\left[1 - e^{-\lambda}\sum_{k=0}^{q}\frac{\lambda^k}{k!}\right]$$

where $\lambda \equiv bM^2$.

Effective hadronization models

$$\langle u^p \rangle = \mathbb{E}[u^p] = \frac{p!}{\prod_{k=2}^{p+1}(a+k)}, \quad \langle v^q \rangle = \mathbb{E}[v^q] = \frac{q!}{\lambda^q(1-e^{-\lambda})}\left[1 - e^{-\lambda}\sum_{k=0}^{q}\frac{\lambda^k}{k!}\right]$$

Note some interesting features of these moments:

- All moments of u are M-independent
- For large λ i.e. M, the v moments are power suppressed

$$\mathbb{E}[v^q] \simeq \frac{q!}{\lambda^q} + \mathcal{O}(e^{-\lambda}).$$

Effective hadronization models – UV

A tower of effective models with dynamics controlled by M!

In the large string mass limit i.e. the UV, all v-moments are power suppressed leaving

$$\mu_{\rm UV} = c_1 M, \quad \sigma^2_{\rm UV} = d_1 M^2$$

Matching power of M from the computation of $\langle \Delta M \rangle = \langle \sqrt{1-u} - 1 \rangle$ gives

$$c_1 = -\frac{1}{2a+3}, \quad d_1 = \frac{a+1}{(a+2)(2a+3)^2}$$

The UV dynamics are scale invariant!

Effective hadronization models – running

A tower of effective models with dynamics controlled by M!

At finite but still large M, the ν-moments induce controlled corrections to the UV fixed point

$$\mu_{\text{int}} = C_1(\lambda)M, \quad \sigma^2_{\text{UV}} = D_1(\lambda)M^2$$

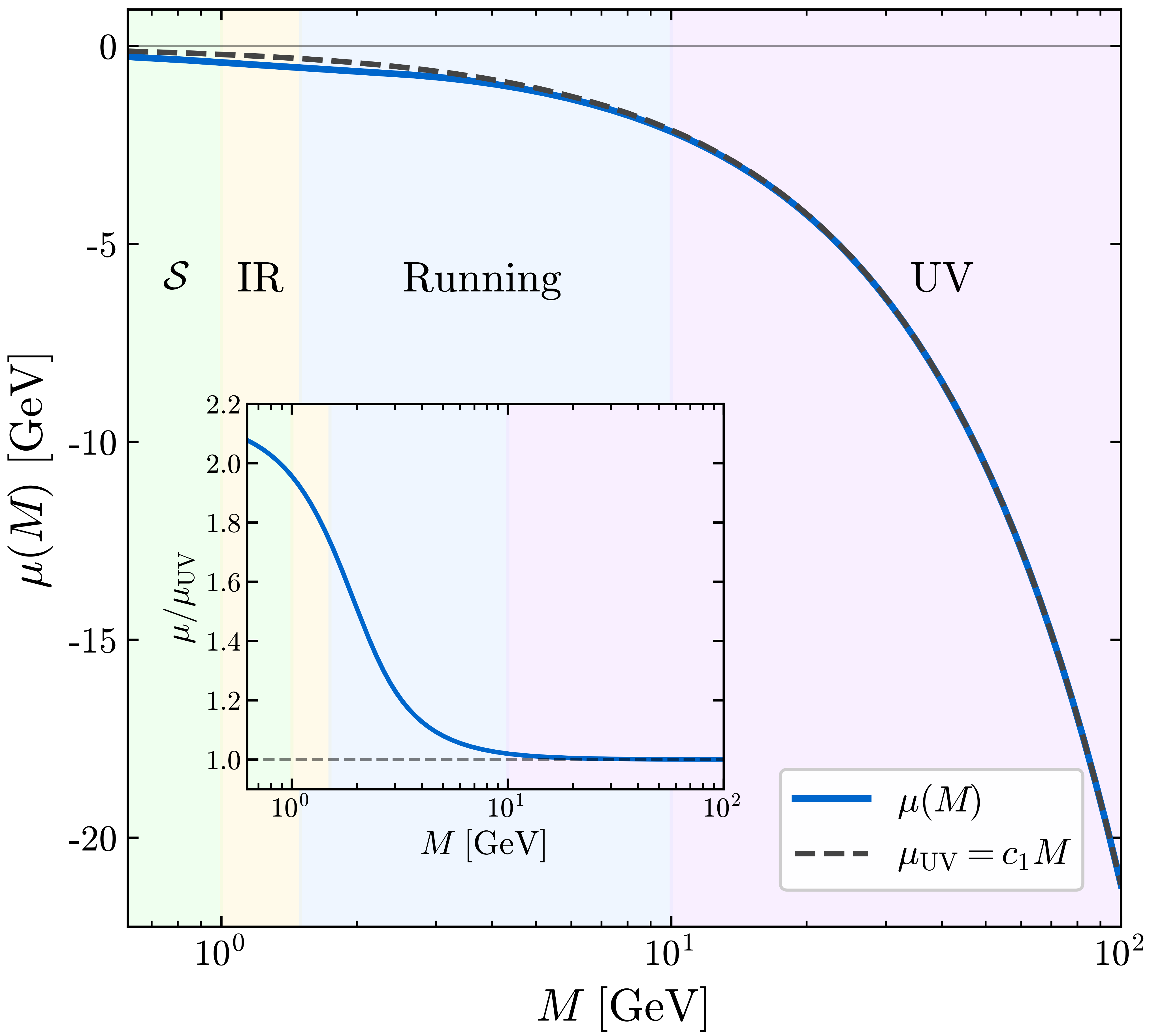
$$C_1(\lambda) = A_1 B_1(\lambda) - 1$$

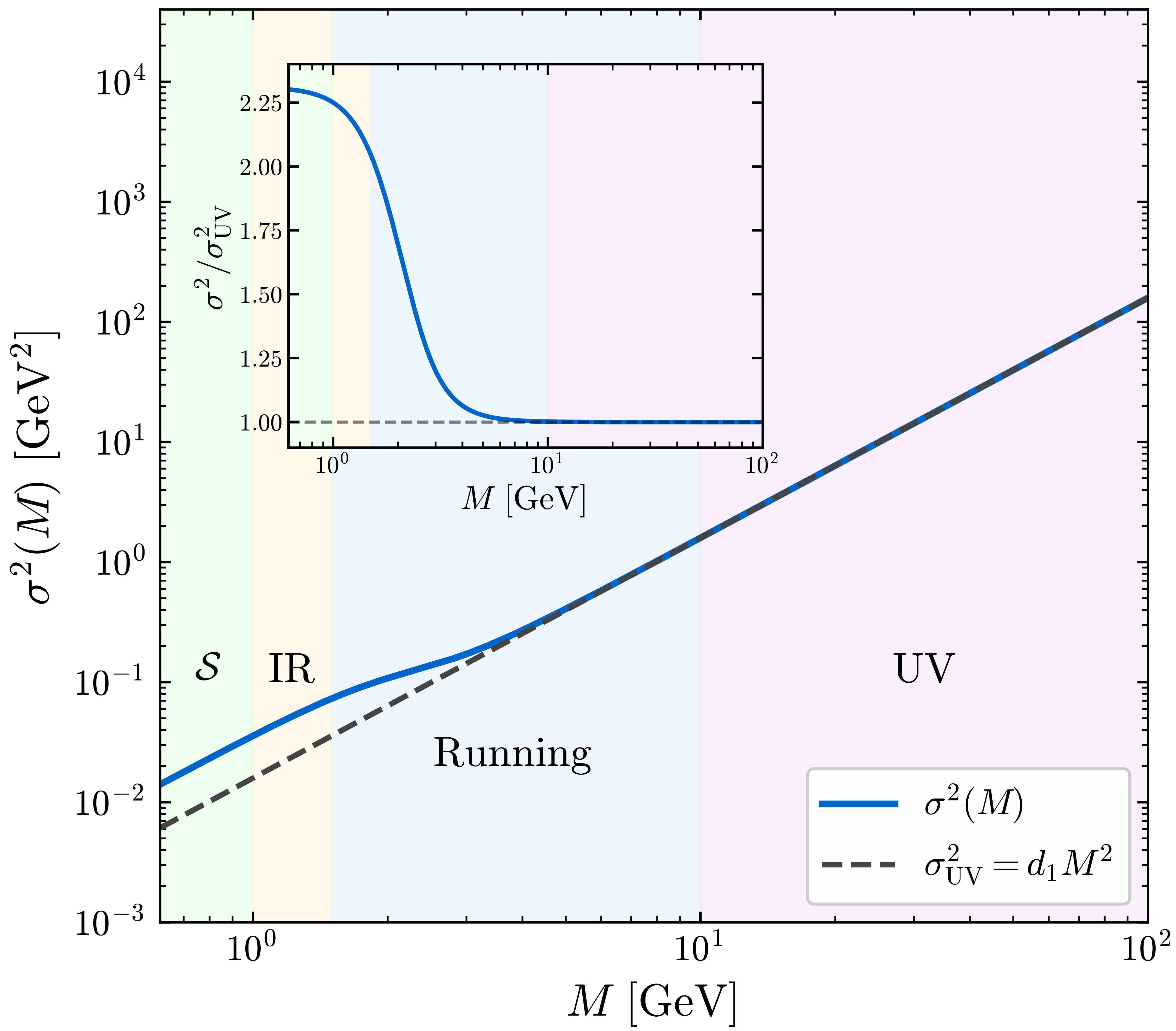
$$D_1(\lambda) = A_2 B_2(\lambda) - (A_1 B_1(\lambda))^2$$

$$A_1 = \frac{a+1}{a+3/2}, \qquad B_1(\lambda) = \frac{1}{1-e^{-\lambda}}\left[1 - \frac{e^{-\lambda}\sqrt{\pi}}{2\sqrt{\lambda}}\text{erfi}(\sqrt{\lambda})\right]$$

$$A_2 = \frac{a+1}{a+2}, \qquad B_2(\lambda) = 1 - \frac{1}{\lambda} + \frac{e^{-\lambda}}{1-e^{-\lambda}}.$$

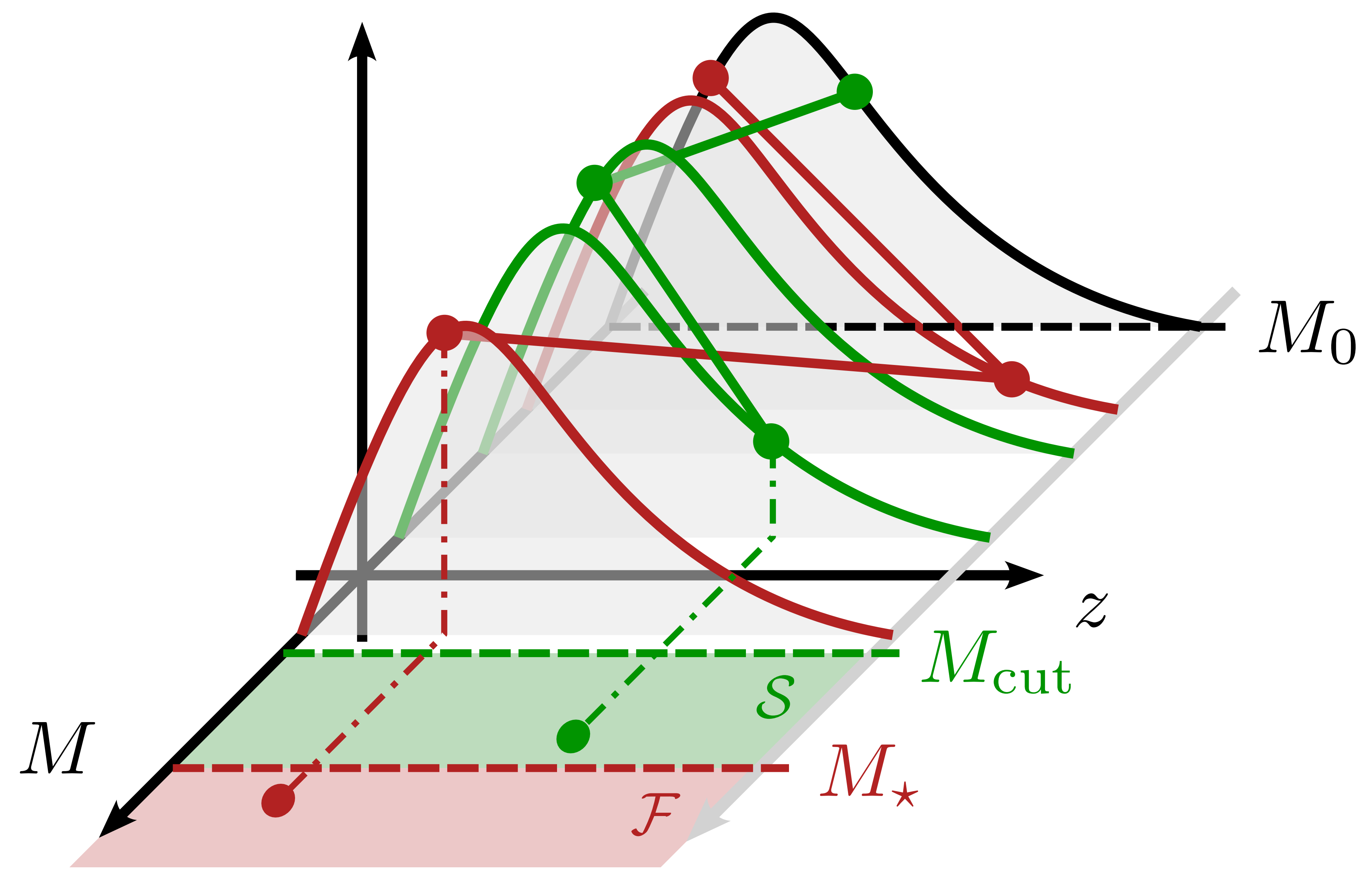
Effective hadronization models





Non-local tail operators

As we get closer to the termination region, it becomes possible for $\mathcal{O}(1)$ jumps coming from rare samples of the “tail” to push us directly into the success or failure regions. To capture these dynamics we must introduce “tail operators”



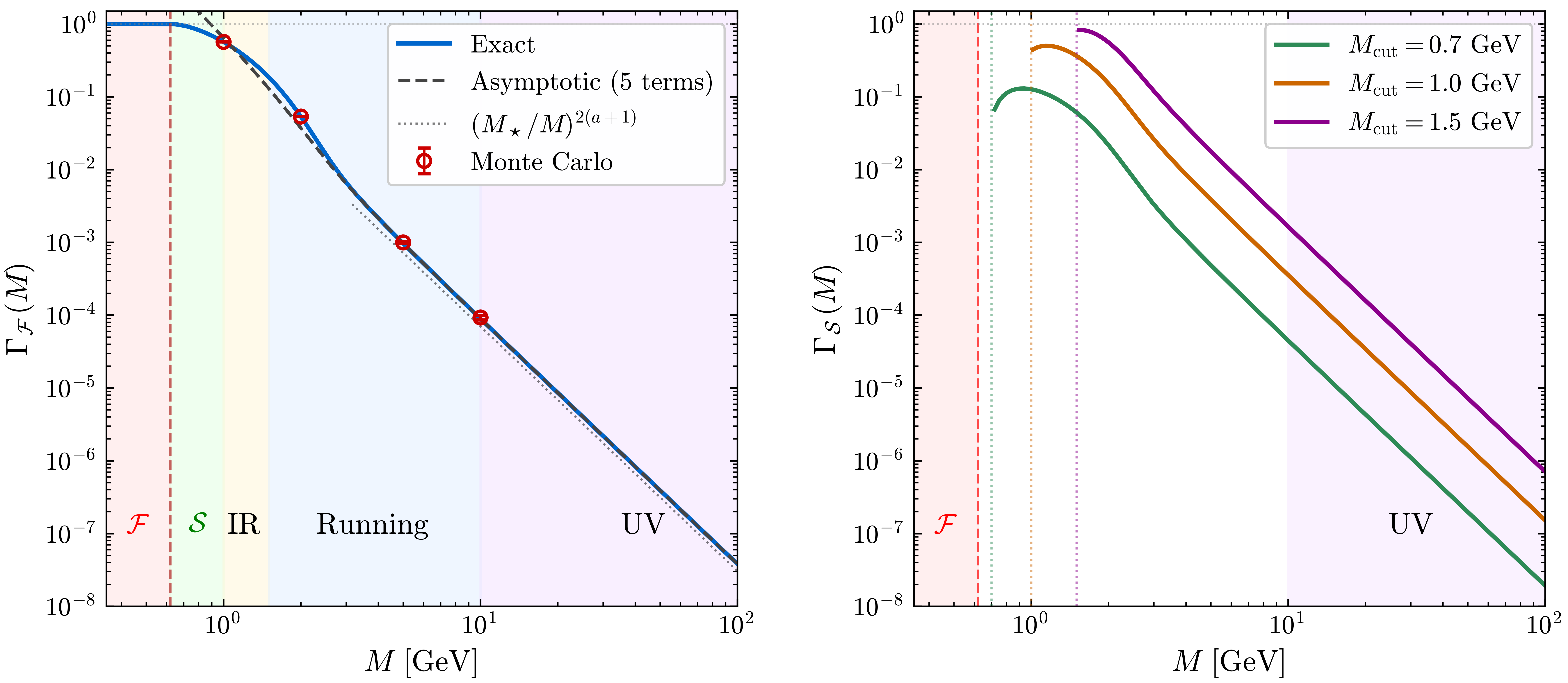
$$\Gamma_{\mathcal{F}}(M) \equiv \mathbb{P}(M_{n+1} < M_\star \mid M_n = M)\,,$$

$$\Gamma_{\mathcal{S}}(M) \equiv \mathbb{P}(M_\star \le M_{n+1} \le M_{\text{cut}} \mid M_n = M)\,,$$

Strongly suppressed for large M

$$\Gamma_{\mathcal{F},\mathcal{S}}(M) \sim \left(\frac{M_{\text{scale}}}{M}\right)^{2(a+1)}$$

Non-local tail operators



The effective generator $\mathscr{L}$

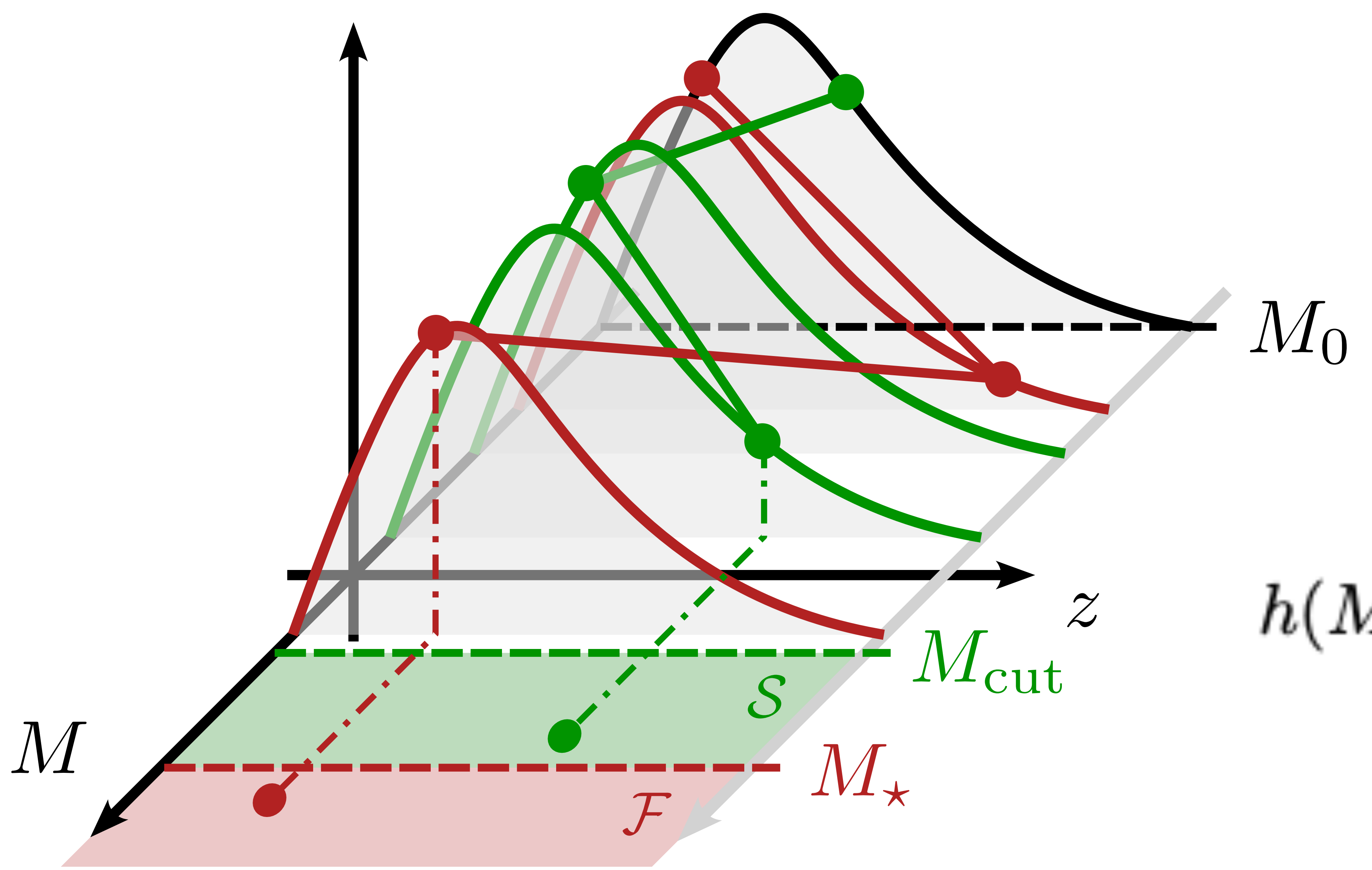
We now have a tower of hadronization dynamics controlled by the string mass M. For a given M, the dynamics and observables are governed by an effective generator

$$\mathscr{L}_{\text{eff}}(M) = \mathscr{L}_{\text{loc}} + \mathscr{L}_{\text{nonloc}}$$

with

$$\mathcal{L}_{\text{nonloc}}\,\mathcal{O}(M) = \Gamma_{\mathcal{F}}(M)\big[\mathcal{O}(M_\star) - \mathcal{O}(M)\big] + \Gamma_{\mathcal{S}}(M)\big[\mathcal{O}(M_{\text{cut}}) - \mathcal{O}(M)\big]$$

The success probability



Similarly to the Brownian motion example, we'd like to understand qualitatively how "efficient" the algorithm is at obeying the constraint.

$$h(M) = \mathbb{P}_{\rm bare}\big(\mathcal{C} \,\big|\, M_0 = M\big) = \mathbb{P}_{\rm bare}\big(\text{reach } M \in \mathcal{S} \text{ before } M \in \mathcal{F} \,\big|\, M_0 = M\big)$$

with boundary conditions

$$h(M) = 1 \quad (M \in \mathcal{S}), \qquad h(M) = 0 \quad (M \in \mathcal{F})$$

The success probability

Again from the example at the beginning, we know h is an invariant of the stochastic evolution i.e.

$$\mathscr{L}_{\text{eff}}\, h = 0.$$

We are left with an ODE

$$\mu(M)\, h'(M) + \tfrac{1}{2}\sigma^2(M)\, h''(M) + \Gamma_{\mathcal{S}}(M)\, [1 - h(M)] - \Gamma_{\mathcal{F}}(M)\, h(M) = 0$$

that can be solved within each EFT explicitly and matched across the thresholds.

The success probability across the EFTs

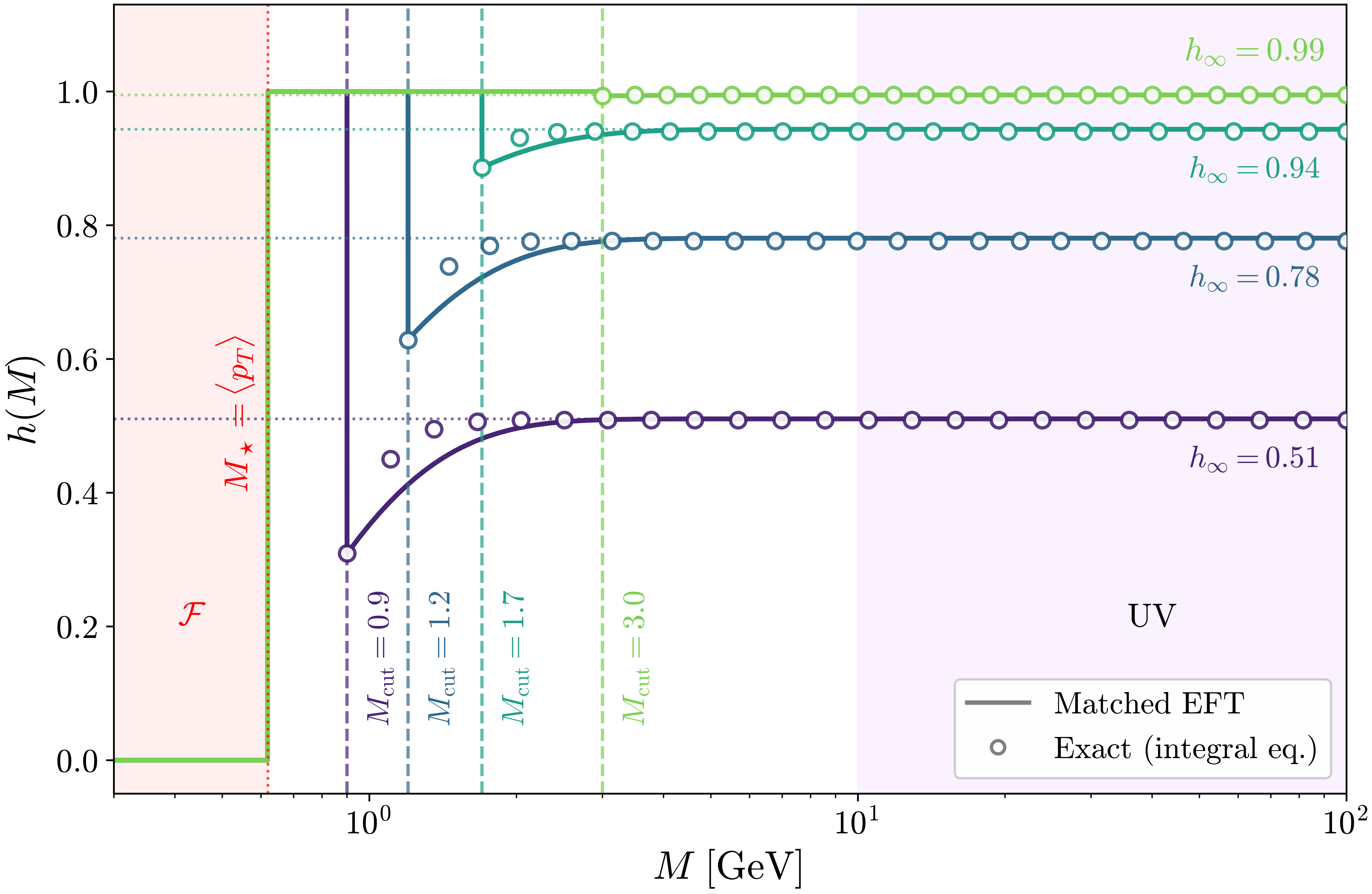
Summarizing:

- The UV solution gives a constant value h_∞, as expected, which depends on the width of the success region
- The intermediate running region also maintains a constant value
- Near the boundary, the non-local operators induce non-trivial dynamics — an exponential rise from the boundary to the asymptotic value

$$h_{\text{bdry}}(M) = h_{\text{loc}}^{\text{cut}} + B\, e^{r_-(M-M_{\text{cut}})}, \qquad r_- < 0.$$

$$r_\pm = \frac{-\mu_{\text{cut}} \pm \sqrt{\mu_{\text{cut}}^2 + 2\sigma_{\text{cut}}^2 \Gamma_{\text{cut}}}}{\sigma_{\text{cut}}^2}$$

The success probability



Conditioning and local renormalization

Now that we have the success probability, we can in turn condition the dynamics on the constraint by renormalizing the drift

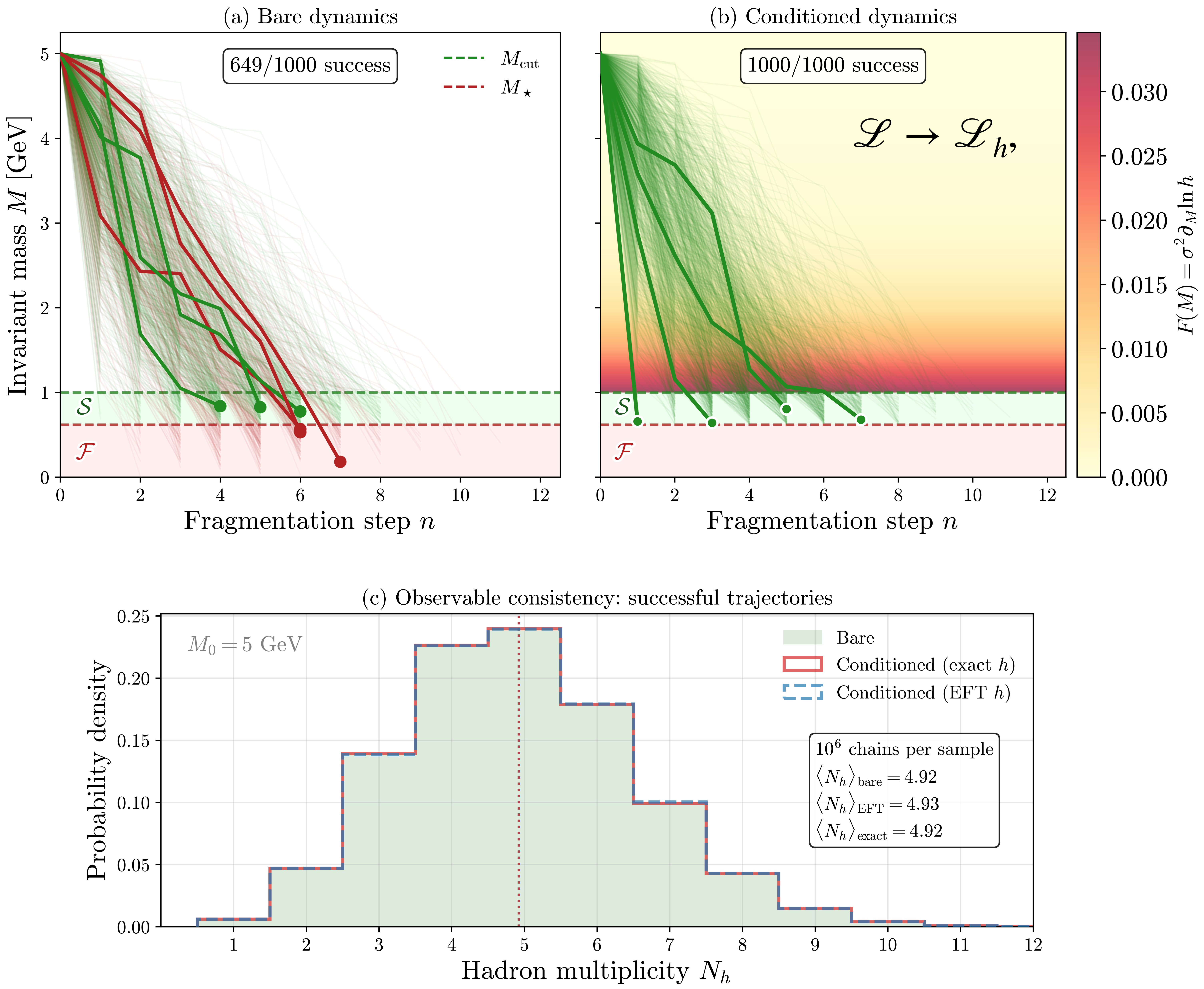
$$\mu_h(M) = \mu(M) + \sigma^2(M)\,\partial_M \ln h(M)$$

This can also be understood as an effective force on the trajectory space

$$F(M) \equiv \sigma^2(M)\,\partial_M \ln h(M) = -\sigma^2(M)\,\partial_M V_{\text{eff}}(M)$$

Markovian restoration

Positive force $F > 0$ i.e. a positive gradient $h' > 0$ indicates a "slowing force"

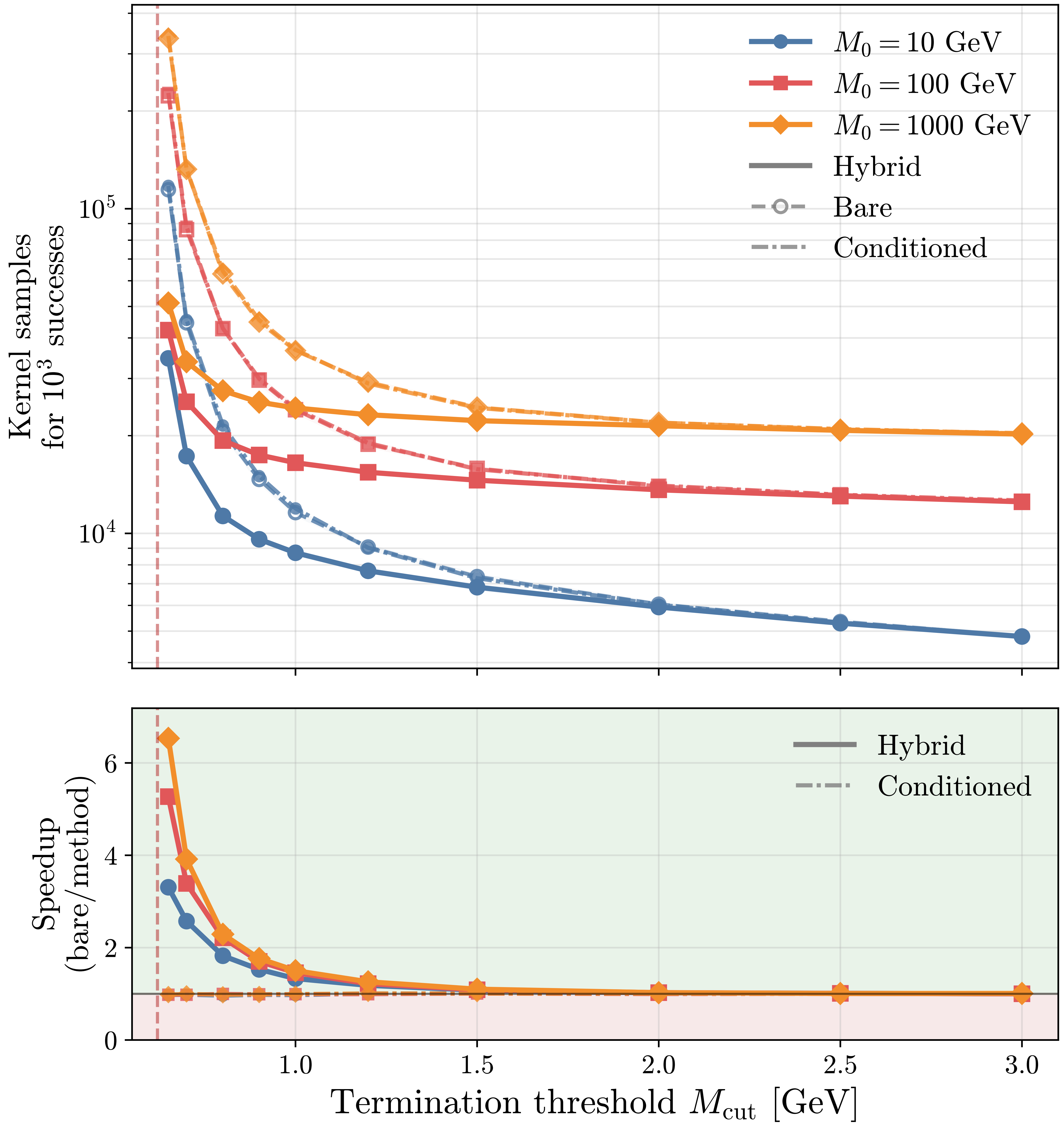


Computational cost

$$C_{\text{bare}} = \frac{L}{h_\infty} = L + L\left(\frac{1}{h_\infty} - 1\right)$$

$$C_{\text{cond}} = \sum_{i=1}^{L} \frac{1}{h(M_i)} = L + \sum_{i=1}^{L}\left(\frac{1}{h(M_i)} - 1\right) \approx L + L\left(\frac{1}{h_\infty} - 1\right)$$

$$C_{\text{hybrid}} = L + \mathcal{O}(1)$$



Conclusions

Hadronization can be reformulated as a conditioned stochastic process. This description admits a tower of effective hadronization models that can be used in turn to compute observables.

Much to do in the future…

- Gluons
- Clean(er)? data-driven extraction
- Finite mass corrections
- Explicit matching to cluster
- Full-fledged implementation?

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Thanks for your attention :)

Joint versus Pythia marginals