SCALAR QED

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1 Scalar QED

Consider the theory of a complex scalar field Φ, describing particles and antiparticles χ^- and χ^+ of electric charge $\mp e$, respectively, interacting with the electromagnetic field A^μ. The Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\Phi)^* D^\mu\Phi - M^2\Phi^*\Phi \tag{1}$$

Where $D_\mu \equiv \partial_\mu + ieA_\mu$.

1.1 Propagator

The free propagator is defined by

$$\begin{aligned} i\Delta(x-y) &\equiv \langle 0|T\{\Phi(x)\Phi^*(y)\}|0\rangle \\ &= \langle 0|\Phi(x)\Phi^*(y)|0\rangle\,\Theta(x_0-y_0) + \langle 0|\Phi^*(y)\Phi(x)|0\rangle\,\Theta(y_0-x_0) \end{aligned} \tag{2}$$

A free real scalar field can be described by the following operator acting on Fock space

$$\phi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}\sqrt{2p^0}}\left[a(\mathbf{p})e^{-ip\cdot x} + a^{c\dagger}(\mathbf{p})e^{ip\cdot x}\right] \tag{3}$$

With the creation and annihilation operators obeying (anti-)commutation relations

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')]_\mp = \delta^3(\mathbf{p}-\mathbf{p}'), \qquad [a(\mathbf{p}), a(\mathbf{p}')]_\mp = [a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')]_\mp = 0 \tag{4}$$

A complex scalar field can be described by two independent real scalar fields

$$\begin{aligned} \Phi(x) &= \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\phi_2(x)\right) \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}\sqrt{2p^0}}\left[a(\mathbf{p})e^{-ip\cdot x} + b^\dagger(\mathbf{p})e^{ip\cdot x}\right] \end{aligned} \tag{5}$$

$$\Phi^*(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}\sqrt{2p^0}}\left[a^\dagger(\mathbf{p})e^{ip\cdot x} + b(\mathbf{p})e^{-ip\cdot x}\right] \tag{6}$$

Where the $a(\mathbf{p})$ operators create and annihilate single scalar particle states with positive charge ($\hat{Q} = +1$) and the $b(\mathbf{p})$ operators create and annihilate single scalar particle states with negative charge ($\hat{Q} = -1$). This can be seen by computing the conserved current $j^\mu = i\Phi^\dagger \overleftrightarrow{\partial^\mu} \Phi$. Where $\overleftrightarrow{\partial^\mu}$ denotes differentiation to the right with a plus sign and to the left with a minus sign. The charge operator $\hat{Q}$ is then given by

$$\hat{Q} = \int d^3\mathbf{x} j^0 = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2p^0} \left[a^\dagger(\mathbf{p})a(\mathbf{p}) - b^\dagger(\mathbf{p})b(\mathbf{p})\right] \tag{7}$$

From the commutations relations of a real scalar field Eq.(4) we have that

$$\begin{aligned}[a(\mathbf{p}), a^\dagger(\mathbf{p}')]_\mp &= \delta^3(\mathbf{p} - \mathbf{p}') \\ [b(\mathbf{p}), b^\dagger(\mathbf{p}')]_\mp &= \delta^3(\mathbf{p} - \mathbf{p}')\end{aligned} \tag{8}$$

Evaluating the propagator explicitly we see

$$\begin{aligned} i\Delta(x-y) = \int & \frac{d^3\mathbf{p}}{(2\pi)^{3/2}\sqrt{2p^0}} \frac{d^3\mathbf{q}}{(2\pi)^{3/2}\sqrt{2q^0}} \\ & \times \Big[\Theta(x_0 - y_0) \langle 0| \left(a(\mathbf{p})e^{-ip\cdot x} + b^\dagger(\mathbf{p})e^{ip\cdot x}\right) \left(a^\dagger(\mathbf{q})e^{ip\cdot y} + b(\mathbf{q})e^{-iq\cdot y}\right) |0\rangle \\ & + \Theta(y_0 - x_0) \langle 0| \left(a^\dagger(\mathbf{q})e^{iq\cdot y} + b(\mathbf{q})e^{-iq\cdot y}\right) \left(a(\mathbf{p})e^{-ip\cdot x} + b^\dagger(\mathbf{p})e^{ip\cdot x}\right) |0\rangle \Big] \end{aligned} \tag{9}$$

$$\begin{aligned} = \int & \frac{d^3\mathbf{p}}{(2\pi)^{3/2}\sqrt{2p^0}} \frac{d^3\mathbf{q}}{(2\pi)^{3/2}\sqrt{2q^0}} \Big[\Theta(x_0 - y_0)e^{i(q\cdot y - p\cdot x)} \langle 0|a(\mathbf{p})a^\dagger(\mathbf{q})|0\rangle \\ & + \Theta(y_0 - x_0)e^{i(p\cdot x - q\cdot y)} \langle 0|b(\mathbf{p})b^\dagger(\mathbf{q})|0\rangle \Big] \end{aligned} \tag{10}$$

$$= \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}\sqrt{2p^0}} \frac{d^3\mathbf{q}}{(2\pi)^{3/2}\sqrt{2q^0}} \delta^3(\mathbf{p} - \mathbf{q}) \left[\Theta(x_0 - y_0)e^{i(q\cdot y - p\cdot x)} + \Theta(y_0 - x_0)e^{i(p\cdot x - q\cdot y)}\right] \tag{11}$$

$$= \int \frac{d^3\mathbf{p}}{(2\pi)^3 2p^0} \left[\Theta(x_0 - y_0)e^{-ip\cdot(x-y)} + \Theta(y_0 - x_0)e^{ip\cdot(x-y)}\right] \tag{12}$$

Using the Fourier representation of the Heaviside-step function

$$\Theta(t) = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} ds \frac{\exp(-ist)}{s + i\epsilon} \tag{13}$$

and evaluating 12 gives

$$= \int d^4p \left[\frac{1}{(2\pi)^4} \frac{i}{p^2 - M^2 + i\epsilon}\right] e^{ip\cdot(x-y)}. \tag{14}$$

1.2 External legs

We note that the creation and annihilation operators can be written in the following form which may prove useful

$$\begin{aligned} b^\dagger(\mathbf{p}) &= \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}\sqrt{2p^0}}\Phi(x)i \overleftrightarrow{\partial_t}\, e^{-ip\cdot x} \\ &= \int \frac{d^3\mathbf{q}}{(2\pi)^3\sqrt{2q^0}\sqrt{2p^0}} \int d^3\mathbf{x} \left\{ \left[a(\mathbf{q})e^{-iq\cdot x} + b^\dagger(\mathbf{q})e^{iq\cdot x}\right] i \overleftrightarrow{\partial_t}\, e^{-ip\cdot x} \right\} \\ &= \int \frac{d^3\mathbf{q}}{(2\pi)^3\sqrt{2q^0}\sqrt{2p^0}} \int d^3\mathbf{x} \left\{ (p^0+q^0)e^{i(q-p)\cdot x}a^\dagger(\mathbf{q}) + (p^0-q^0)e^{-i(p+q)\cdot x}a(\mathbf{q}) \right\} \end{aligned} \tag{15}$$

Using

$$\int d^3\mathbf{x} e^{i(q-p)\cdot x} = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}) \tag{16}$$

And then integrating over the delta-function

$$= \frac{1}{(2\pi)^3 2p^0}(2\pi)^3 2p^0 b^\dagger(\mathbf{p}) = b^\dagger(\mathbf{p}) \tag{17}$$

Similarly,

$$a^\dagger(\mathbf{p}) = \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}\sqrt{2p^0}}\Phi^*(x)i \overleftrightarrow{\partial_t}\, e^{-ip\cdot x} \tag{18}$$

These don't seem to be of any help here but, the explicit form is still interesting and could be useful.

The external legs contain contractions of the form

$$\begin{aligned} \langle 0|\, \Phi(x)b^\dagger(\mathbf{k})\, |0\rangle &= |0\rangle \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}\sqrt{2p^0}} \left[a(\mathbf{p})e^{-ip\cdot x} + b^\dagger(\mathbf{p})e^{ip\cdot x}\right] b^\dagger(\mathbf{k}) + \cdots |0\rangle \\ &= \left[\frac{1}{(2\pi)^{3/2}\sqrt{2p^0}}\right] \int d^3\mathbf{p} e^{ip\cdot x}\delta^4(\mathbf{p}-\mathbf{k})\, \langle 0| \cdots |0\rangle \\ &= \left[\frac{1}{(2\pi)^{3/2}\sqrt{2p^0}}\right] e^{ik\cdot x}\, \langle 0| \cdots |0\rangle \end{aligned} \tag{19}$$

Where I have used the derived scalar commutation given in Eq.(8). All other external legs will follow the same general calculation, always leaving a factor of $1/(2\pi)^{3/2}\sqrt{2p^0}$ as the coefficient.

1.3 Feynman Rules

The above external states are contracted with other fields when we evaluate the time-ordered product, we obtain factors of the form

$$\langle 0|\, a \tag{20}$$

All of the information about interactions between the scalar particles and photons is held in the covariant derivative term in the Lagrangian density

$$(D_\mu \Phi(x))^* D^\mu \Phi(x) = (\partial_\mu \Phi^*(x) - ieA_\mu(y)\Phi^*(x))(\partial^\mu \Phi(x) + ieA^\mu(x)\Phi(x)) \tag{21}$$

$$= \partial_\mu \Phi^*(x)\partial^\mu \Phi(x) + ieA^\mu(x)\Phi(x)\partial_\mu \Phi^*(x) - ieA_\mu(x)\Phi^*(x)\partial^\mu \Phi(x) + e^2 A_\mu(x)A^\mu(x)|\Phi(x)|^2$$

The first term is just the free kinetic term for the complex scalar field. Fourier transforming to momentum space and focusing on the second and third term

$$\begin{aligned} &ie\int d^4x\,[A_\mu(x)\Phi(x)\partial^\mu\Phi^*(x) - A_\mu(x)\Phi^*(x)\partial^\mu\Phi(x)] \\ &\rightsquigarrow i^2 e\int d^4x \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi^4)}\left[-A_\mu(k)\Phi(p)p^{\mu\prime}\Phi^*(p') + A_\mu(k)\Phi^*(p')p^\mu\Phi(p)\right] e^{-ix\cdot(p+k-p')} \\ &= \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4} A_\mu(k)\Phi^*(p')\Phi(p)e(p'-p)^\mu(2\pi)^4\delta^4(p-p'+k) \end{aligned} \tag{22}$$

The Feynman rules are obtained from the time ordered products of fields from a perturbation expansion of $e^{i\mathcal{L}_{\rm int}}$. Thus, we see that the above term corresponds to the vertex $ie(p'-p)^\mu$ with external lines corresponding to Φ, Φ^*, and A_μ with an overall energy-momentum conserving delta-function $(2\pi)^4\delta^4(p+p'-k)$. Pictorially this is represented as

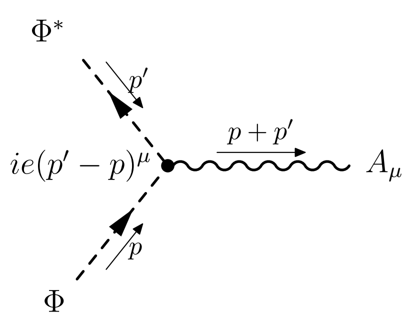


Figure 1: Scalar QED vertex

For the final term in Eq.(21) we have

$$
\begin{aligned}
&e^2 \int d^4x \left[g_{\mu\nu} A^\nu(x) A^\mu(x) |\Phi(x)|^2 \right] \\
&\quad \rightsquigarrow \int d^4x \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \frac{d^4p'}{(2\pi)^4} \left[e^2 g_{\mu\nu} A^\nu(k) A^\mu(k') \Phi(p) \Phi^*(p') e^{-ix\cdot(p+p'-k-k')} \right]
\end{aligned} \tag{23}
$$

$$
= \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \frac{d^4p'}{(2\pi)^4} \left[A^\nu(k) A^\mu(k') \Phi(p) \Phi^*(p') e^2 g_{\mu\nu} (2\pi)^4 \delta^4(p + p' - k - k') \right] \tag{24}
$$

Including a factor of i from the expansion of $e^{i\mathcal{L}_{\text{int}}}$ we have a four-particle interaction vertex of with a vertex factor $2ie^2(2\pi)^4\delta^4(p+p'-k)$. The factor of 2 comes from the necessary symmetry factor which accounts for the total number of A_μ contractions which result in the same amplitude. Pictorially this vertex is represented as

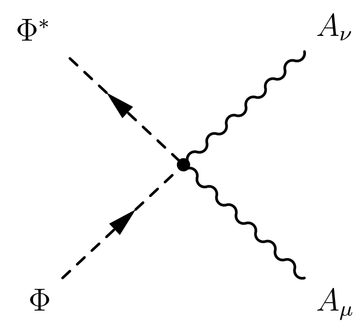


Figure 2: Scalar QED vertex

1.4 $e^+e^- \to \chi^+\chi^-$ scattering

The lowest order tree-level for $e^+e^- \to \chi^+\chi^-$ is given by

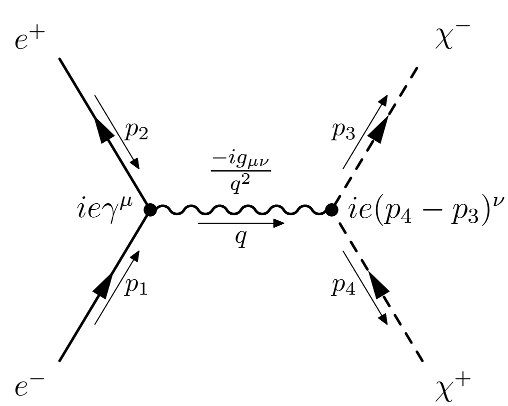


Figure 3: s-channel Feynman diagram

Our Feynman rules give the transition amplitude:

$$-2\pi i\delta^4(p_1+p_2-p_3-p_4)\mathcal{M} =$$
$$\int \frac{d^4q}{(2\pi)^4}\left[\frac{\bar{v}(\mathbf{p}_2,\sigma_2)}{(2\pi)^{3/2}} ie\gamma^\mu \frac{u(\mathbf{p}_1,\sigma_1)}{(2\pi)^{3/2}} \frac{1}{(2\pi)^4}\frac{-ig_{\mu\nu}}{q^2} ie(p_4-p_3)^\nu \frac{1}{(2\pi)^{3/2}\sqrt{2p_3^0}}\frac{1}{(2\pi)^{3/2}\sqrt{2p_4^0}}\right]$$
$$\times (2\pi)^4\delta^4(p_1+p_2-q)(2\pi)^4\delta^4(q-p_3-p_4)$$
$$= \frac{ie^2}{2(2\pi)^2\sqrt{p_3^0p_4^0}}\left[\bar{v}_2\frac{\not{p}_4-\not{p}_3}{(p_1+p_2)^2}u_1\right]\delta^4(p_1+p_2-p_3-p_4) \tag{25}$$

Leaving us with our matrix element

$$\mathcal{M} = \frac{-e^2}{2(2\pi)^3\sqrt{p_3^0p_4^0}}\left[\bar{v}_2\frac{\not{p}_4-\not{p}_3}{(p_1+p_2)^2}u_1\right] \tag{26}$$

The squared matrix element is then given by

$$|\mathcal{M}|^2 = \frac{e^4}{4(2\pi)^6p_3^0p_4^0}\left[\bar{v}_2\frac{\not{p}_4-\not{p}_3}{(p_1+p_2)^2}u_1\right]\left[\bar{v}_2\frac{\not{p}_4-\not{p}_3}{(p_1+p_2)^2}u_1\right]^* \tag{27}$$
$$= \frac{e^4}{4(2\pi)^6p_3^0p_4^0}\left[\bar{v}_2\frac{\not{p}_4-\not{p}_3}{(p_1+p_2)^2}u_1\right]\left[\bar{u}_1\frac{\not{p}_4-\not{p}_3}{(p_1+p_2)^2}v_2\right] \tag{28}$$

Now we want to average over the initial helicites

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4(2\pi)^6p_3^0p_4^0}\frac{1}{4}\sum_{\sigma_1,\sigma_2}\left[\bar{v}_2\frac{\not{p}_4-\not{p}_3}{(p_1+p_2)^2}u_1\right]\left[\bar{u}_1\frac{\not{p}_4-\not{p}_3}{(p_1+p_2)^2}v_2\right] \tag{29}$$

Using our completeness relation for spinors Eq.(5.7) in the lecture notes

$$= \frac{e^4}{32(2\pi)^6p_1^0p_3^0p_4^0}\sum_{i,j}\left[\frac{\not{p}_4-\not{p}_3}{(p_1+p_2)^2}(\not{p}_1+m_e)\frac{\not{p}_4-\not{p}_3}{(p_1+p_2)^2}\right]_{ij}\sum_{\sigma_2}[v_2\bar{v}_2]_{ji} \tag{30}$$

$$= \frac{e^4}{64(2\pi)^6p_1^0p_2^0p_3^0p_4^0}\frac{1}{(p_1+p_2)^4}\mathrm{Tr}[(\not{p}_4-\not{p}_3)(\not{p}_1+m_e)(\not{p}_4-\not{p}_3)(\not{p}_2-m_e)] \tag{31}$$

Ignoring the electron mass we are left with

$$= \frac{e^4}{64(2\pi)^6 p_1^0 p_2^0 p_3^0 p_4^0} \frac{1}{(p_1+p_2)^4} \mathrm{Tr}[(\not{p}_4 - \not{p}_3)\not{p}_1(\not{p}_4 - \not{p}_3)\not{p}_2] \tag{32}$$

$$= \frac{e^4}{16(2\pi)^6 p_1^0 p_2^0 p_3^0 p_4^0} \frac{1}{(p_1+p_2)^4} \left[2\left(p_2 \cdot p_3 - p_2 \cdot p_4\right)\left(p_1 \cdot p_3 - p_1 \cdot p_4\right) - (p_1 \cdot p_2)(p_3 - p_4)^2 \right] \tag{33}$$

In the center of mass frame $\mathbf{p_1} = -\mathbf{p_2} \equiv \mathbf{p}$, $p_1 + p_2 = (2E_{\mathbf{p}}, \mathbf{0})$ thus, $p_3 + p_4 = (2E_p, \mathbf{p})$. Definine $E_{\mathbf{p}} \equiv E$ and assuming $E \gg m_e$ i.e. $|\mathbf{p}| \approx E$

$$p_1 = (E, \mathbf{p}), \qquad p_2 = (E, -\mathbf{p}) \tag{34}$$

$$p_3 = (E', \mathbf{p}'), \qquad p_4 = (E', -\mathbf{p}') \tag{35}$$

Thus,

$$(p_2 \cdot p_3) = (p_1 \cdot p_4) = E^2 + \mathbf{p} \cdot \mathbf{p}' = E^2 + |\mathbf{p}||\mathbf{p}'| \cos\theta \approx E(E + |\mathbf{p}'| \cos\theta) \tag{36}$$

$$(p_1 \cdot p_3) = (p_2 \cdot p_4) = E^2 - \mathbf{p} \cdot \mathbf{p}' = E^2 - |\mathbf{p}||\mathbf{p}'| \cos\theta \approx E(E - |\mathbf{p}'| \cos\theta) \tag{37}$$

$$(p_1 \cdot p_2) = E^2 + |\mathbf{p}|^2 \approx 2E^2 \tag{38}$$

$$(p_3 \cdot p_4) = E^2 + |\mathbf{p}'|^2 \tag{39}$$

Substituting these expressions into Eq.(33) gives

$$= \frac{e^4}{2(2\pi)^6 E^4} \frac{1}{16E^2} \left[E^2 |\mathbf{p}'|^2 (1 - \cos^2\theta) \right] \tag{40}$$

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4 |\mathbf{p}'|^2}{32(2\pi)^6 E^6} \sin^2\theta \tag{41}$$

From Eq.(2.55) in the lecture notes the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{(2\pi)^4 |\mathbf{p}'| E_1 E_2 E_3 E_4}{(E_1 + E_2)^2 |\mathbf{p}|} \langle |\mathcal{M}|^2 \rangle = \frac{(2\pi)^4 |\mathbf{p}'| E^4}{4} \langle |\mathcal{M}|^2 \rangle = \frac{e^4 |\mathbf{p}'|^3}{128(2\pi)^2 E^5} \sin^2\theta \tag{42}$$

Defining $s \equiv (2E)^2$ and $\alpha = e^2/(4\pi)$

$$\frac{d\sigma(e^+e^- \to \chi^+\chi^-)}{d\Omega} = \frac{\alpha^2}{8s} \frac{(E^2 - m_\chi^2)^{3/2}}{E^3} \sin^2\theta = \frac{\alpha^2}{8s} \left(1 - \frac{m_\chi^2}{E^2}\right)^{3/2} \sin^2\theta \tag{43}$$

The total cross section is given by

$$\sigma(e^+e^- \to \chi^+\chi^-) = \frac{2\pi\alpha^2}{8s}\left(1 - \frac{m_\chi^2}{E^2}\right)^{3/2}\int_{-1}^{1}(1-\cos^2\theta)d\cos\theta \tag{44}$$

$$= \frac{\pi\alpha^2}{3s}\left(1 - \frac{m_\chi^2}{E^2}\right)^{3/2} \tag{45}$$

In the high-energy limit ($E \gg m_\chi$) we have

$$\frac{d\sigma(e^+e^- \to \chi^+\chi^-)}{d\Omega} \rightsquigarrow \frac{\alpha^2}{8s}\sin^2\theta \tag{46}$$

$$\sigma(e^+e^- \to \chi^+\chi^-) \rightsquigarrow \frac{\pi\alpha^2}{3s} \tag{47}$$

The differential cross section for $e^+ + e^- \to \mu^+ + \mu^-$ is given by Eq.(7.48) in the lecture notes

$$\frac{d\sigma(e^+e^- \to \mu^+\mu^-)}{d\Omega} = \frac{\alpha^2}{4s}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left[1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right] \tag{48}$$

And the total cross section (Eq.(7.49))

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left(1 + \frac{1}{2}\frac{m_\mu^2}{E^2}\right) \tag{49}$$

In the high-energy limit ($E \gg m_\mu$)

$$\frac{d\sigma(e^+e^- \to \mu^+\mu^-)}{d\Omega} \rightsquigarrow \frac{\alpha^2}{4s}(1+\cos^2\theta) \tag{50}$$

$$\sigma(e^+e^- \to \mu^+\mu^-) \rightsquigarrow \frac{4\pi\alpha^2}{3s} \tag{51}$$

Interestingly the total cross sections have the same dependence on α and only differ by a factor of four.

1.5 Anomalous magnetic moment contribution

Now assume that there is an electrically neutral χ^0 particles, described by a real scalar field ϕ, that interacts with electrons via a Yukawa interaction, given by

$$\mathcal{L}_Y = -\lambda\phi\bar{\psi}_e\psi_e \tag{52}$$

Where λ is a real constant and ψ_e is the electron field. Find the Feynman rule for the Yukawa interaction and calculate the effect of virtual χ^0 particles to the anomalous magnetic moment of the electron to one-loop accuracy.

Transforming to momentum space we have

$$-\lambda\int d^4x\phi(x)\bar{\psi}_e(x)\psi_e(x) \rightsquigarrow -\lambda\int d^4x\int\frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\phi(k)\bar{\psi}_e(p')\psi_e(p)e^{-ix\cdot(p-p'+k)} \tag{53}$$

$$= \int\frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\phi(k)\bar{\psi}_e(p')\psi_e(p)(-\lambda)(2\pi)^4\delta^4(p+k-p') \tag{54}$$

Thus, we have a vertex factor of $-i\lambda(2\pi)^4\delta^4(p+k-p')$ represented pictorially as the following Feynman diagram

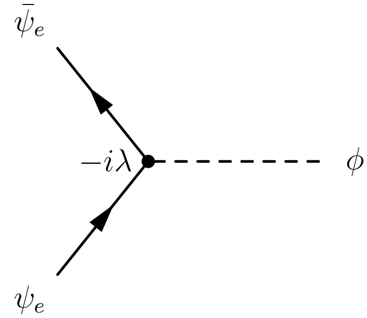
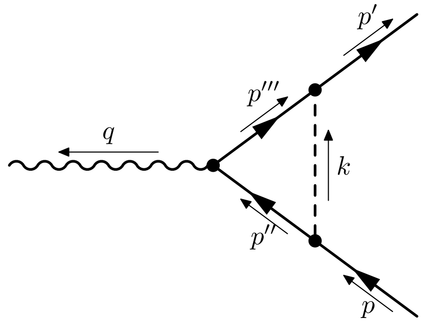


Figure 4: Neutral scalar vertex

The contribution to the anomalous magnetic moment can be calculated from analyzing the following Feynman diagram



Writing the one-loop matrix element and ignoring the external factors as in the notes

$$
\begin{aligned}
&-ie(2\pi)^4\Gamma^\mu(p,p') \\
&\quad = \int d^4k d^4p'' d^4p''' (-i\lambda)(2\pi)^4 \frac{i}{(2\pi)^4}\frac{\not{p}''' + m}{(p''')^2 - m^2 + i\epsilon}(-ie)(2\pi)^4\gamma^\mu \\
&\quad \times \frac{i}{(2\pi)^4}\frac{\not{p}'' + m}{(p')^2 - m^2 + i\epsilon}(-i\lambda)(2\pi)^4\frac{1}{(2\pi)^4}\frac{i}{k^2 - M^2 + i\epsilon} \\
&\quad \times \delta^4(p - k - p'')\delta^4(p''' + k - p')
\end{aligned} \tag{55}
$$

$$
= \lambda^2 e \int d^4k \frac{\not{p}' - \not{k} + m}{(p' - k)^2 - m^2 + i\epsilon}\gamma^\mu \frac{\not{p} - \not{k} + m}{(p - k)^2 - m^2 + i\epsilon}\frac{1}{k^2 - M^2 + i\epsilon} \tag{56}
$$

$$
\Gamma^\mu = i\lambda^2 \int \frac{d^4k}{(2\pi)^4}\frac{(\not{p}' - \not{k} + m)\gamma^\mu(\not{p} - \not{k} + m)}{[(p' - k)^2 - m^2][(p - k)^2 - m^2][k^2 - M^2]} \tag{57}
$$

Using the identity from problem set 8

$$
\frac{1}{ABC} = 2\int_0^1 \delta(x + y + z - 1)\frac{dxdydz}{[xA + yB + zC]^3} \tag{58}
$$

Eq.(57) becomes

$$
= 2\int_0^1 dxdydz\delta^3(x+y+z-1)\left[[(p' - k)^2 - m^2]x + [(p - k)^2 - m^2]y + [k^2 - M^2]z\right]^{-3} \tag{59}
$$

$$
= 2\int_0^1 dxdydz\delta^3(x+y+z-1)\left[k^2(x + y + z) - 2k\cdot(py + p'x) - m^2(x + y) - M^2z + p^2y + p'^2x\right]^{-3} \tag{60}
$$

Completing the square

$$= 2\int_0^1 dxdydz\delta^3(x+y+z-1)\left[(k-px-p'y))^2-(px+p'y)^2-M^2z\right]^{-3} \tag{61}$$

Defining $k \to k+px+p'y$, $\Lambda \equiv (px+p'y)^2+M^2z$

$$= 2\int_0^1 dxdydz\delta^3(x+y+z-1)\left[k^2-\Lambda\right]^{-3} \tag{62}$$

The numerator is given by

$$\begin{aligned}&(\not{p}'-\not{k}+m)\gamma^\mu(\not{p}-\not{k}+m)\\ &\quad\rightsquigarrow \left[\not{p}'(1-y)-\not{p}x-\not{k}+m\right]\gamma^\mu\left[\not{p}(1-x)-\not{p}'y-\not{k}+m\right]\end{aligned} \tag{63}$$

Because the denominator is even in k' we can use the fact that $k'^\mu k'^\nu = g^{\mu\nu}k'^2/4$ and drop terms odd in k' leaving us with

$$\left[\not{p}'(1-y)-\not{p}x+m\right]\gamma^\mu\left[\not{p}(1-x)-\not{p}'y+m\right]+\not{k}'\gamma^\mu\not{k}' \tag{64}$$

$$\begin{aligned}=&\frac{k^2}{4}\gamma_\lambda\gamma^\mu\gamma^\lambda+(1-y)(1-x)\not{p}'\gamma^\mu\not{p}-(1-y)y\not{p}'\gamma^\mu\not{p}'+(1-y)m\not{p}'\gamma^\mu\\ &-(1-x)x\not{p}\gamma^\mu\not{p}+xy\not{p}\gamma^\mu\not{p}'-xm\not{p}\gamma^\mu\\ &+(1-x)m\gamma^\mu\not{p}-ym\gamma^\mu\not{p}'+m^2\gamma^\mu\end{aligned} \tag{65}$$

We can now use the Dirac equation and Clifford relation to simplify, using

$$\not{p}\gamma^\mu = 2p^\mu-\gamma^\mu\not{p},\quad \not{p}u(p)=u(p)m,\quad \bar{u}(p')\not{p}'=\bar{u}(p')m,\quad q^2=-2(p\cdot p')+2m^2 \tag{66}$$

1.
$$\frac{k'^2}{4}\gamma_\lambda\gamma^\mu\gamma^\lambda = \frac{-k'^2}{2}\gamma^\mu \tag{67}$$

2.
$$(1-y)(1-x)\not{p}'\gamma^\mu\not{p} = (1-x)(1-y)m^2\gamma^\mu \tag{68}$$

3.
$$(1-y)y\not{p}'\gamma^\mu\not{p}' = (1-y)ym\gamma^\mu\not{p}' = (1-y)ym(2p'^\mu-\not{p}'\gamma^\mu) \tag{69}$$
$$= (1-y)ym(2p'^\mu-m\gamma^\mu) \tag{70}$$

4.
$$(1-y)m\not{p}'\gamma^\mu = (1-y)m^2\gamma^\mu \tag{71}$$

5.
$$(1-x)x\not{p}\gamma^\mu\not{p} = (1-x)xm(2p^\mu - \gamma^\mu\not{p}) = (1-x)xm(2p^\mu - m\gamma^\mu) \tag{72}$$

6.
$$xy\not{p}\gamma^\mu\not{p}' = xy(2p^\mu - \gamma^\mu\not{p})\not{p}' \tag{73}$$
$$= xym\left[2p^\mu - (2p'^\mu - \not{p}'\gamma^\mu)\right] = xym(2q^\mu + \gamma^\mu m) \tag{74}$$

7.
$$xm\not{p}\gamma^\mu = xm(2p^\mu - \gamma^\mu m) \tag{75}$$

8.
$$(1-x)m\gamma^\mu\not{p} = (1-x)m^2\gamma^\mu \tag{76}$$

9.
$$ym\gamma^\mu\not{p}' = ym(2p'^\mu - \gamma^\mu m) \tag{77}$$

10.
$$m^2\gamma^\mu \tag{78}$$

We are uninterested in terms which are proportional to γ^μ as they do not contribute to the magnetic moment. Thus, putting it all together and excluding those terms we have
$$\rightsquigarrow 2m(p^\mu(x-2)x + q^\mu xy + p'^\mu(y-2)y) \tag{79}$$
$$= 2m\left[p^\mu x^2 + q^\mu xy + p'^\mu y^2 - 2(p^\mu x + p'^\mu y)\right] \tag{80}$$

I can rewrite
$$2(p^\mu x^2 + p'^\mu y^2) = (p^\mu + p'^\mu)(x^2+y^2) + q^\mu(x^2-y^2) \tag{81}$$
$$2(p^\mu x + p'^\mu y) = (p^\mu + p'^\mu)(x+y) + q^\mu(x-y) \tag{82}$$

Giving,
$$= m\left[(p^\mu + p'^\mu)(x^2+y^2) - 2(p^\mu + p'^\mu)(x+y) + q^\mu(x^2-y^2) - 2q^\mu(x-y) + q^\mu xy\right] \tag{83}$$
$$= m\left[(p^\mu + p'^\mu)(x^2+y^2-2x-2y) + q^\mu(x^2-y^2-2x+2y+xy)\right] \tag{84}$$
$$= 2m\left[p'^\mu y(y-x-2) + p^\mu x(y+x-2)\right] \tag{85}$$
$$= m\left[(p^\mu + p'^\mu)(y(y-x-2) + x(y+x-2)) + q^\mu(y(y-x-2) - x(y+x-2))\right] \tag{86}$$

The only factors which contribute to the magnetic moment are proportional to $(p^\mu + p'^\mu)$(because $F(0) = 1$). Now we use the Gordon identity

$$\bar{u}(\mathbf{p}')\gamma^\mu u(\mathbf{p}) = \bar{u}(\mathbf{p}')\left[\frac{p'^\mu + p^\mu}{2m} - \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(\mathbf{p}) \tag{87}$$

Where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. We can ignore γ^μ factor receive the contribution of the numerator as

$$\frac{i\sigma^{\mu\nu}q_\nu}{2m}2m^2(y(y-x-2)+x(y+x-2)) \tag{88}$$

Thus,

$$G(q^2) = 2i\lambda^2 m^2 \int_0^\infty \frac{dk}{(2\pi)^4}\int_0^1 dxdydz\delta^3(x+y+z-1)\frac{k^3(y(y-x-2)+x(y+x-2))}{(k^2-\Lambda)^3} \tag{89}$$

$$G(q^2) = -\frac{2\lambda^2 m}{4(2\pi)^2}\int_0^1 dxdydz\delta^3(x+y+z-1)\frac{(y(y-x-2)+x(y+x-2))}{\Lambda} \tag{90}$$

Rewriting

$$\Lambda = (px+p'y)^2+M^2z = m^2(x^2+y^2)+2xy(p\cdot p')+M^2z = m^2(x^2+y^2)+xy(2m^2-q^2)+M^2z \tag{91}$$

$$= m^2(x^2+y^2+2xy) - xyq^2 + M^2z \tag{92}$$

Leaving us with

$$G(q^2) = -\frac{2\lambda^2 m}{4(2\pi)^2}\int_0^1 dxdydz\delta(x+y+z-1)\frac{(y(y-x-2)+x(y+x-2))}{m^2(x^2+y^2+2xy)-xyq^2+M^2z} \tag{93}$$

At $q^2 = 0$ we have then,

$$G(0) = -\frac{2\lambda^2 m}{4(2\pi)^2}\int_0^1 dxdydz\delta(x+y+z-1)\frac{(y(y-x-2)+x(y+x-2))}{m^2(x^2+y^2+2xy)-xyq^2+M^2z} \tag{94}$$

Integrating over x invokes the delta-function $(x \to 1-y-z)$.

$$= \frac{\lambda^2 m^2}{2(2\pi)^2}\int_0^1 dz\int_0^{1-z} dy\frac{1-z^2}{m^2(1-z)^2+M^2z} \tag{95}$$

$$= \frac{\lambda^2 m^2}{2(2\pi)^2}\int_0^1 dz\frac{(1-z^2)(1-z)}{m^2(1-z)^2+M^2z} \tag{96}$$

This integral can be evaluated in Mathematica assuming $M \gg m$, we end up with the neutral scalars contribution to the electrons magnetic moment as

$$G(0) \approx \frac{\lambda^2 m^2}{8\pi^2 M^2}\left[\ln\left(\frac{M^2}{m^2}\right) - \frac{7}{6}\right] \tag{97}$$