MØLLER SCATTERING

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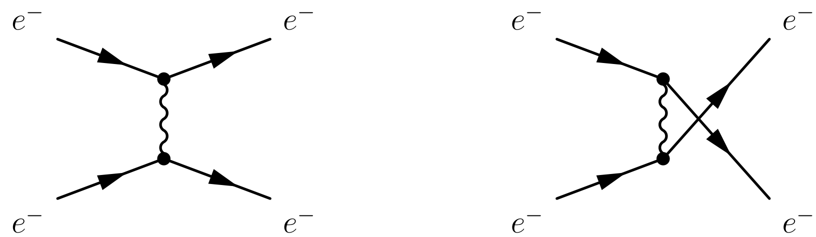
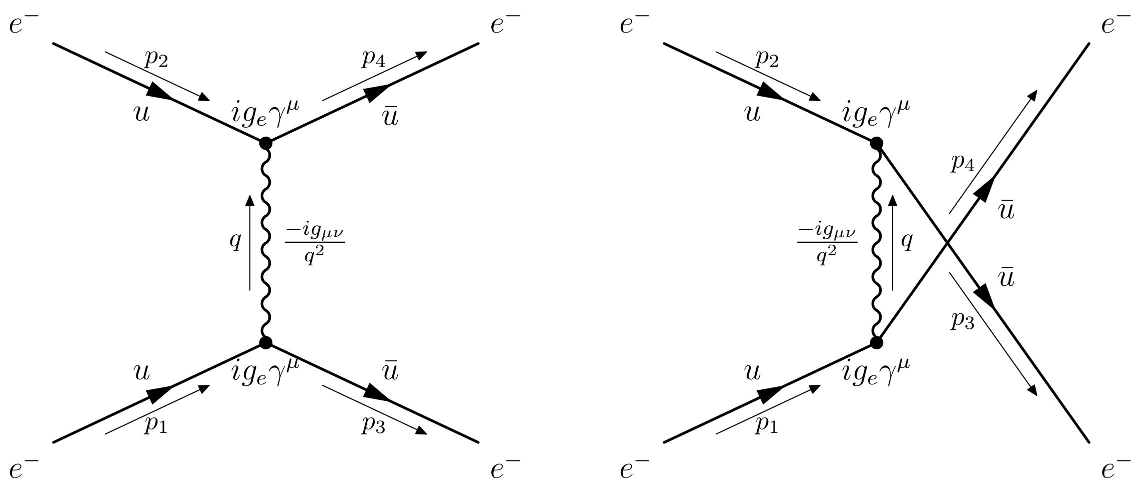
1 Møller Scattering

Møller scattering or electron-electron scattering has contributions from u and t-channel diagrams



In more detail



The Feynman rules for the t-channel yield

$$
\begin{aligned}
\mathcal{M}_t = \int \frac{d^4q}{(2\pi)^4}[\bar{u}(p_4)ig_e\gamma^\mu u(p_2)]\frac{-ig_{\mu\nu}}{q^2}[\bar{u}(p_3)ig_e\gamma^\nu u(p_1)] \\
(2\pi)^8\delta^4(p_2+q-p_4)\delta^4(p_1-q-p_3)
\end{aligned} \tag{1}
$$

$$
= -\frac{g_e^2}{(p_1-p_3)^2}[\bar{u}(p_4)\gamma^\mu u(p_2)][\bar{u}(p_3)\gamma_\mu u(p_1)] \tag{2}
$$

For the u-channel we have

$$\mathcal{M}_u = \int \frac{d^4q}{(2\pi)^4}[\bar{u}(p_4)ig_e\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\bar{u}(p_3)ig_e\gamma^\nu u(p_2)] \\ (2\pi)^8\delta^4(p_2+q-p_3)\delta^4(p_1-q-p_4) \tag{3}$$

$$= -\frac{g_e^2}{(p_1-p_4)^2}[\bar{u}(p_4)\gamma^\mu u(p_1)][\bar{u}(p_3)\gamma_\mu u(p_2)] \tag{4}$$

Antisymmeterizing the sum of the amplitudes yields the total amplitude

$$\mathcal{M}_{\text{tot}} = -\frac{g_e^2}{(p_1-p_3)^2}[\bar{u}(p_4)\gamma^\mu u(p_2)][\bar{u}(p_3)\gamma_\mu u(p_1)]+\frac{g_e^2}{(p_1-p_4)^2}[\bar{u}(p_4)\gamma^\mu u(p_1)][\bar{u}(p_3)\gamma_\mu u(p_2)]$$

Defining $t \equiv p_1 - p_3$ and $u \equiv p_1 - p_4$ and switching to a more convinient notation for the spinors $u\left(p_{1,2,3,4}\right) = u_{1,2,3,4}$. Squaring the amplitude yields

$$|\mathcal{M}_{\text{tot}}|^2 = \left(-\frac{g_e^2}{t^2}[\bar{u}_4\gamma^\mu u_2][\bar{u}_3\gamma_\mu u_1] + \frac{g_e^2}{u^2}[\bar{u}_4\gamma^\mu u_1][\bar{u}_3\gamma_\mu u_2]\right) \\ \times\left(-\frac{g_e^2}{t^2}[\bar{u}_4\gamma^\mu u_2]^*[\bar{u}_3\gamma_\mu u_1]^* + \frac{g_e^2}{u^2}[\bar{u}_4\gamma^\mu u_1]^*[\bar{u}_3\gamma_\mu u_2]^*\right) \tag{5}$$

$$=g_e^4\left(\frac{1}{t^4}[\bar{u}_4\gamma^\mu u_2][\bar{u}_3\gamma_\mu u_1][\bar{u}_4\gamma^\nu u_2]^*[\bar{u}_3\gamma_\nu u_1]^* - \frac{1}{t^2u^2}[\bar{u}_4\gamma^\mu u_2][\bar{u}_3\gamma_\mu u_1][\bar{u}_4\gamma^\nu u_1]^*[\bar{u}_3\gamma_\nu u_2]^* \right. \\ \left. - \frac{1}{t^2u^2}[\bar{u}_4\gamma^\mu u_1][\bar{u}_3\gamma_\mu u_2][\bar{u}_4\gamma^\nu u_2]^*[\bar{u}_3\gamma_\nu u_1]^* + \frac{1}{u^4}[\bar{u}_4\gamma^\mu u_1][\bar{u}_3\gamma_\mu u_2][\bar{u}_4\gamma^\nu u_1]^*[\bar{u}_3\gamma_\nu u_2]^*\right)$$

Noting,

$$[\bar{u}_4\gamma^\nu u_1]^* = [u_4^\dagger\gamma^0\gamma^\nu u_1]^\dagger = [u_1^\dagger\gamma^{\nu\dagger}\gamma^{0\dagger}u_4] \tag{6}$$

$\gamma^{\nu\dagger} = \gamma^0\gamma^\nu\gamma^0$, $\gamma^{0\dagger} = \gamma^0$, and $(\gamma^0)^2 = \mathbb{1}$

$$= [u_1\gamma^0\gamma^\nu\gamma^0\gamma^0 u_4] = [\bar{u}_1\gamma^\nu u_4] \tag{7}$$

Likewise,

$$[\bar{u}_3\gamma^\nu u_2]^* = [\bar{u}_2\gamma^\nu u_3] \tag{8}$$

Rewriting Eq.(1)

$$=g_e^4\left(\frac{1}{t^4}[\bar{u}_4\gamma^\mu u_2][\bar{u}_3\gamma_\mu u_1][\bar{u}_2\gamma^\nu u_4][\bar{u}_1\gamma_\nu u_3] - \frac{1}{t^2u^2}[\bar{u}_4\gamma^\mu u_2][\bar{u}_3\gamma_\mu u_1][\bar{u}_1\gamma^\nu u_4][\bar{u}_2\gamma_\nu u_3] \right. \\ \left. - \frac{1}{t^2u^2}[\bar{u}_4\gamma^\mu u_1][\bar{u}_3\gamma_\mu u_2][\bar{u}_2\gamma^\nu u_4][\bar{u}_1\gamma_\nu u_3] + \frac{1}{u^4}[\bar{u}_4\gamma^\mu u_1][\bar{u}_3\gamma_\mu u_2][\bar{u}_1\gamma^\nu u_4][\bar{u}_2\gamma_\nu u_3]\right)$$

Remembering the completeness relation for spin 1/2 particles

$$\sum_{s=1,2} u^{(s)}\bar{u}^{(s)} = (\gamma^\mu p_\mu + mc) \tag{9}$$

Focusing one term at a time as we begin to sum over the initial spins

$$[\bar{u}_4\gamma^\mu u_2][\bar{u}_3\gamma_\mu u_1][\bar{u}_2\gamma^\nu u_4][\bar{u}_1\gamma_\nu u_3] = [\bar{u}_4\gamma^\mu u_2][\bar{u}_2\gamma^\nu u_4][\bar{u}_3\gamma_\mu u_1][\bar{u}_1\gamma_\nu u_3] \tag{10}$$

First averaging over the spins of particles one and two

$$\frac{1}{2}\sum_{s_1=1,2}\frac{1}{2}\sum_{s_2=1,2}\bar{u}_4\gamma^\mu[u_2^{(s_2)}\bar{u}_2^{(s_2)}]\gamma^\nu u_4\bar{u}_3\gamma_\mu[u_1^{(s_1)}\bar{u}_1^{(s_1)}]\gamma_\nu u_3 \tag{11}$$

$$= \frac{1}{4}\bar{u}_4\gamma^\mu(\not{p}_2 + m_e c)\gamma^\nu u_4\bar{u}_3\gamma_\mu(\not{p}_1 + m_e c)\gamma_\nu u_3 \tag{12}$$

$$= \frac{1}{4}\sum_{i,j,l,m=1}^{4}[\gamma^\mu(\not{p}_2 + m_e c)\gamma^\nu]_{ij}[\gamma_\mu(\not{p}_1 + m_e c)\gamma_\nu]_{lm}[\bar{u}_4 u_4]_{ji}[\bar{u}_3 u_3]_{ml} \tag{13}$$

Summing over the spins of particles 3 and 4 now

$$\frac{1}{4}\sum_{i,j,l,m=1}^{4}[\gamma^\mu(\not{p}_2+m_e c)\gamma^\nu]_{ij}[\gamma_\mu(\not{p}_1+m_e c)\gamma_\nu]_{lm}\sum_{s_4=1,2}[\bar{u}_4^{(s_4)}u_4^{(s_4)}]_{ji}\sum_{s_3=1,2}[\bar{u}_3^{(s_3)}u_3^{(s_3)}]_{ml} \tag{14}$$

$$= \frac{1}{4}\sum_{i,j,l,m=1}^{4}[\gamma^\mu(\not{p}_2 + m_e c)\gamma^\nu]_{ij}[\gamma_\mu(\not{p}_1 + m_e c)\gamma_\nu]_{lm}[(\not{p}_4 + m_e c)]_{ji}[(\not{p}_3 + m_e c)]_{ml} \tag{15}$$

$$= \frac{1}{4}\sum_{i,l=1}^{4}[\gamma^\mu(\not{p}_2 + m_e c)\gamma^\nu(\not{p}_4 + m_e c)]_{ii}[\gamma_\mu(\not{p}_1 + m_e c)\gamma_\nu(\not{p}_3 + m_e c)]_{ll} \tag{16}$$

$$= \frac{1}{4}\mathrm{Tr}[\gamma^\mu(\not{p}_2 + m_e c)\gamma^\nu(\not{p}_4 + m_e c)]\mathrm{Tr}[\gamma_\mu(\not{p}_1 + m_e c)\gamma_\nu(\not{p}_3 + m_e c)] \tag{17}$$

Now we need to evaluate the traces, ignoring the small mass of the electron we are left with

$$= \frac{1}{4}\mathrm{Tr}[\gamma^\mu\not{p}_2\gamma^\nu\not{p}_4]\mathrm{Tr}[\gamma_\mu\not{p}_1\gamma_\nu\not{p}_3] = \frac{1}{4}p_{2\sigma}p_{4\lambda}{p_1}^\xi{p_3}^\rho\mathrm{Tr}[\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\lambda]\mathrm{Tr}[\gamma_\mu\gamma_\xi\gamma_\nu\gamma_\rho] \tag{18}$$

Using the identity $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})$

$$= \frac{1}{4}p_{2\sigma}p_{4\lambda}{p_1}^\xi{p_3}^\rho\left[4(g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\nu}g^{\sigma\lambda} + g^{\mu\lambda}g^{\sigma\nu})4(g_{\mu\xi}g_{\nu\rho} - g_{\mu\nu}g_{\xi\rho} + g_{\mu\rho}g_{\xi\nu})\right] \tag{19}$$

$$= 4p_{2\sigma}p_{4\lambda}(g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\nu}g^{\sigma\lambda} + g^{\mu\lambda}g^{\sigma\nu}){p_1}^{\xi}{p_3}^{\rho}(g_{\mu\xi}g_{\nu\rho} - g_{\mu\nu}g_{\xi\rho} + g_{\mu\rho}g_{\xi\nu}) \tag{20}$$

$$= 4\left((-g^{\mu\nu}(p_2\cdot p_4) + p_2^{\mu}p_4^{\nu} + p_2^{\nu}p_4^{\mu})(-g_{\mu\nu}(p_1\cdot p_3) + p_{1\nu}p_{3\mu} + p_{1\mu}p_{3\nu})\right) \tag{21}$$

$$\begin{aligned}=&4\big(4(p_2\cdot p_4)(p_1\cdot p_3) - (p_2\cdot p_4)(p_1\cdot p_3) - (p_2\cdot p_4)(p_1\cdot p_3) - (p_2\cdot p_4)(p_1\cdot p_3)\\ &+ (p_1\cdot p_4)\,(p_2\cdot p_3) + (p_2\cdot p_1)(p_4\cdot p_3) - (p_2\cdot p_4)(p_1\cdot p_3) + (p_2\cdot p_1)(p_4\cdot p_3)\\ &+ (p_1\cdot p_4)(p_2\cdot p_3)\big)\end{aligned} \tag{22}$$

$$= 8\left((p_1\cdot p_4)(p_2\cdot p_3) + (p_2\cdot p_1)(p_4\cdot p_3)\right) \tag{23}$$

The last term in Eq.(1) is obtained in a very similar matter with $p_3 \leftrightarrow p_4$

$$[\bar{u}_4\gamma^{\mu}u_1][\bar{u}_3\gamma_{\mu}u_2][\bar{u}_1\gamma^{\nu}u_4][\bar{u}_2\gamma_{\nu}u_3] = 8\left((p_2\cdot p_4)(p_1\cdot p_3) + (p_1\cdot p_2)(p_4\cdot p_3)\right) \tag{24}$$

Now we look at the cross terms,

$$[\bar{u}_4\gamma^{\mu}u_2][\bar{u}_3\gamma_{\mu}u_1][\bar{u}_1\gamma^{\nu}u_4][\bar{u}_2\gamma_{\nu}u_3] = [\bar{u}_4\gamma^{\mu}u_2][\bar{u}_2\gamma_{\nu}u_3][\bar{u}_3\gamma_{\mu}u_1][\bar{u}_1\gamma^{\nu}u_4] \tag{25}$$

Averaging over spins for particles one and two first yields

$$\frac{1}{4}\bar{u}_4\gamma^{\mu}(\not{p}_2 + m_e c)\gamma^{\nu}u_3\bar{u}_3\gamma_{\mu}(\not{p}_1 + m_e c)\gamma_{\nu}u_4 \tag{26}$$

Summing over particle three

$$= \frac{1}{4}\bar{u}_4\gamma^{\mu}(\not{p}_2 + m_e c)\gamma^{\nu}(\not{p}_3 + m_e c)\gamma_{\mu}(\not{p}_1 + m_e c)\gamma_{\nu}u_4 \tag{27}$$

$$\frac{1}{4}\sum_{i,j=1}^{4}[\gamma^{\mu}(\not{p}_2 + m_e c)\gamma^{\nu}(\not{p}_3 + m_e c)\gamma_{\mu}(\not{p}_1 + m_e c)\gamma_{\nu}]_{ij}[\bar{u}_4 u_4]_{ji} \tag{28}$$

Summing over s_4

$$\frac{1}{4}\sum_{i,j=1}^{4}[\gamma^{\mu}(\not{p}_2 + m_e c)\gamma^{\nu}(\not{p}_3 + m_e c)\gamma_{\mu}(\not{p}_1 + m_e c)\gamma_{\nu}]_{ij}\sum_{s_4=1,2}[\bar{u}_4^{(s_4)}u_4^{(s_4)}]_{ji} \tag{29}$$

$$= \frac{1}{4}\sum_{i,j=1}^{4}[\gamma^{\mu}(\not{p}_2 + m_e c)\gamma^{\nu}(\not{p}_3 + m_e c)\gamma_{\mu}(\not{p}_1 + m_e c)\gamma_{\nu}]_{ij}[(\not{p}_4 + m_e c)]_{ji} \tag{30}$$

$$= \frac{1}{4}\sum_{i=1}^{4}[\gamma^{\mu}(\not{p}_2 + m_e c)\gamma^{\nu}(\not{p}_3 + m_e c)\gamma_{\mu}(\not{p}_1 + m_e c)\gamma_{\nu}(\not{p}_4 + m_e c)]_{ii} \tag{31}$$

$$= \frac{1}{4}\mathrm{Tr}[\gamma^\mu(\not{p}_2 + m_e c)\gamma^\nu(\not{p}_3 + m_e c)\gamma_\mu(\not{p}_1 + m_e c)\gamma_\nu(\not{p}_4 + m_e c)] \tag{32}$$

Ignoring the mass of the electron again

$$\approx \frac{1}{4}\mathrm{Tr}[\gamma^\mu \not{p}_2 \gamma^\nu \not{p}_3 \gamma_\mu \not{p}_1 \gamma_\nu \not{p}_4] \tag{33}$$

$$= \frac{1}{4} p_{2\sigma} p_{3\lambda} p_{1\xi} p_{4\rho} \mathrm{Tr}[\gamma^\mu \gamma^\sigma \gamma^\nu \gamma^\lambda \gamma_\mu \gamma^\xi \gamma_\nu \gamma^\rho] \tag{34}$$

$$= \frac{1}{4} p_{2\sigma} p_{3\lambda} p_{1\xi} p_{4\rho} \mathrm{Tr}[\gamma^\rho \gamma^\mu \gamma^\sigma \gamma^\nu \gamma^\lambda \gamma_\mu \gamma^\xi \gamma_\nu] \tag{35}$$

Using the identity $\mathrm{Tr}[\gamma^\rho \gamma^\mu \gamma^\sigma \gamma^\nu \gamma^\lambda \gamma_\mu \gamma^\xi \gamma_\nu] = -32 g^{\rho\lambda} g^{\sigma\xi}$

$$= -8 p_{2\sigma} p_{3\lambda} p_{1\xi} p_{4\rho} g^{\rho\lambda} g^{\sigma\xi} \tag{36}$$

$$= -8(p_3 \cdot p_4)(p_2 \cdot p_1) \tag{37}$$

Likewise, the other cross-term also gives

$$[\bar{u}_4 \gamma^\mu u_1][\bar{u}_3 \gamma_\mu u_2][\bar{u}_2 \gamma^\nu u_4][\bar{u}_1 \gamma_\nu u_3] = -8(p_3 \cdot p_4)(p_2 \cdot p_1) \tag{38}$$

Putting it all together we have

$$\begin{aligned} \langle|\mathcal{M}_{\text{tot}}|^2\rangle =& 8g_e^4 \Bigg(\frac{1}{t^4}((p_1 \cdot p_4)(p_2 \cdot p_3) + (p_2 \cdot p_1)(p_4 \cdot p_3)) + \frac{2}{t^2 u^2}(p_3 \cdot p_4)(p_2 \cdot p_1) \\ &+ \frac{1}{u^4}\left((p_2 \cdot p_4)(p_1 \cdot p_3) + (p_1 \cdot p_2)(p_4 \cdot p_3)\right) \Bigg) \end{aligned} \tag{39}$$

$$= 8g_e^4 \left(\frac{1}{t^4}(p_1 \cdot p_4)(p_2 \cdot p_3) + \frac{1}{u^4}(p_2 \cdot p_4)(p_1 \cdot p_3) + (p_2 \cdot p_1)(p_4 \cdot p_3) \left(\frac{1}{t^4} + \frac{2}{t^2 u^2} + \frac{1}{u^4} \right) \right) \tag{40}$$

In the approximation where the electron is massless we have

$$t^2 = (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3 = -2p_1 \cdot p_3 \tag{41}$$

$$u^2 = (p_1 - p_4)^2 = p_1^2 + p_4^2 - 2p_1 \cdot p_4 = -2p_1 \cdot p_4 \tag{42}$$

From energy-momentum conservation $p_1 + p_2 = p_3 + p_4$

$$(p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = 2p_1 \cdot p_2 \tag{43}$$

$$(p_3 + p_4)^2 = p_3^2 + p_4^2 + 2p_3 \cdot p_4 = 2p_3 \cdot p_4 \tag{44}$$

Implying that $p_1 \cdot p_2 = p_3 \cdot p_4$. It also implies $p_1 \cdot p_3 = p_2 \cdot p_4$ and $p_1 \cdot p_4 = p_2 \cdot p_3$. Thus,

$$\frac{1}{t^4} + \frac{2}{t^2 u^2} + \frac{1}{u^4} = \frac{1}{4(p_1 \cdot p_3)^2} + \frac{2}{4(p_1 \cdot p_3)(p_1 \cdot p_4)} + \frac{1}{4(p_1 \cdot p_4)^2} \tag{45}$$

$$= \frac{(p_1 \cdot p_4)^2 + 2(p_1 \cdot p_3)(p_1 \cdot p_4) + (p_1 \cdot p_3)^2}{4(p_1 \cdot p_3)^2 (p_1 \cdot p_4)^2} = \frac{(p_1 \cdot p_3 + p_1 \cdot p_4)^2}{4(p_1 \cdot p_3)^2 (p_1 \cdot p_4)^2} \tag{46}$$

Plugging these relations back into Eq.(40)

$$= 8g_e^4 \left(\frac{(p_1 \cdot p_4)(p_2 \cdot p_3)}{4(p_1 \cdot p_3)^2} + \frac{(p_2 \cdot p_4)(p_1 \cdot p_3)}{4(p_1 \cdot p_4)^2} + (p_1 \cdot p_2)(p_4 \cdot p_3) \frac{(p_1 \cdot p_3 + p_1 \cdot p_4)^2}{4(p_1 \cdot p_3)^2 (p_1 \cdot p_4)^2} \right) \tag{47}$$

$$= 2g_e^4 \left(\frac{(p_1 \cdot p_4)^2}{(p_1 \cdot p_3)^2} + \frac{(p_1 \cdot p_3)^2}{(p_1 \cdot p_4)^2} + (p_1 \cdot p_2)^2 \frac{\big(p_1 \cdot (p_3 + p_4)\big)^2}{(p_1 \cdot p_3)^2 (p_1 \cdot p_4)^2} \right) \tag{48}$$

Focusing on the numerator of the last term

$$\big(p_1 \cdot (p_3 + p_4)\big)^2 = \big(p_1 \cdot (p_1 + p_2)\big)^2 = \big(p_1^2 + (p_1 \cdot p_2)\big)^2 = (p_1 \cdot p_2)^2 \tag{49}$$

Putting it all together

$$= 2g_e^4 \left(\frac{(p_1 \cdot p_4)^2}{(p_1 \cdot p_3)^2} + \frac{(p_1 \cdot p_3)^2}{(p_1 \cdot p_4)^2} + \frac{(p_1 \cdot p_2)^4}{(p_1 \cdot p_3)^2 (p_1 \cdot p_4)^2} \right) \tag{50}$$

$$= \frac{2g_e^4}{(p_1 \cdot p_3)^2 (p_1 \cdot p_4)^2} \left((p_1 \cdot p_4)^4 + (p_1 \cdot p_3)^4 + (p_1 \cdot p_2)^4 \right) \tag{51}$$

Rewriting for clarity

$$\langle |\mathcal{M}_{\text{tot}}|^2 \rangle = \frac{2g_e^4}{(p_1 \cdot p_3)^2 (p_1 \cdot p_4)^2} \left[(p_1 \cdot p_4)^4 + (p_1 \cdot p_3)^4 + (p_1 \cdot p_2)^4 \right] \tag{52}$$

At last we arrive at our desired result. It is straightforward from here to calculate the differential or total cross-section by selecting four-momenta in some frame, recomputing $\langle |\mathcal{M}_{\text{tot}}|^2 \rangle$ with these momenta and then placing the squared matrix element into our expression for the total cross section.