γ MATRICES

October 7, 2023

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1 Identities

Use the anticommutation relation $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ and the cyclic property of traces ($\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB) = \mathrm{Tr}(BCA)$) to prove the following

1.1 $(\gamma^\mu)^n = 0$ for $n = 2k+1$ with $k \in \mathbb{Z}^+$

First note the following property of the gamma matrices

$$\mathrm{Tr}(\gamma^\mu) = 0 \tag{1}$$

The first non-trivial case is the product of three gamma matrices

$$\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}) \tag{2}$$

$$= \mathrm{Tr}(2g^{\mu_1\mu_2}\mathbf{1}_{4\times4}\gamma^{\mu_3} - \gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_3}) \tag{3}$$

$$= 2g^{\mu_1\mu_2}\mathrm{Tr}(\gamma^{\mu_3}) - \mathrm{Tr}(\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_3}) \tag{4}$$

$$= -\mathrm{Tr}(\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_3}) = -\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}) \tag{5}$$

Which implies $\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}) = 0$.

Generalizing to the product of n gamma matrices with n being some odd number

$$\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\cdots\gamma^{\mu_n}) \tag{6}$$

We can follow the same procedure

$$= 2g^{\mu_1\mu_2}\mathrm{Tr}(\gamma^{\mu_3}\gamma^{\mu_4}\cdots\gamma^{\mu_n}) - \mathrm{Tr}(\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_3}\cdots\gamma^{\mu_n}) \tag{7}$$

$$= 2g^{\mu_1\mu_2}\left[2g^{\mu_3\mu_4}\mathrm{Tr}(\gamma^{\mu_5}\gamma^{\mu_6}\cdots\gamma^{\mu_n}) - \mathrm{Tr}(\gamma^{\mu_4}\gamma^{\mu_3}\gamma^{\mu_5}\cdots\gamma^{\mu_n})\right] - \mathrm{Tr}(\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_3}\cdots\gamma^{\mu_n}) \tag{8}$$

We can continue this process for each of the traces within the square brackets but because n is odd in each of these terms we can continue to do this until we are left a sum of $\mathrm{Tr}(\gamma^{\mu_n})$ so the terms within the square brackets are exactly 0. Thus, we are left with

$$= -\mathrm{Tr}(\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_3}\cdots\gamma^{\mu_n}) = -\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\cdots\gamma^{\mu_n}) \tag{9}$$

Implying that $\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\cdots\gamma^{\mu_n}) = 0$

A slicker proof utilizes $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$

1.2 $\mathbf{Tr}(\not{a}\not{b}) = 4a \cdot b$

First noting

$$(\gamma^0)^2 = \mathbf{1}_{4\times4}, \qquad (\gamma^k)^2 = -\mathbf{1}_{4\times4} \qquad \text{for } k = 1,2,3 \tag{10}$$

$$\mathrm{Tr}(\not{a}\not{b}) = \mathrm{Tr}(a^\mu\gamma_\mu b^\nu\gamma_\nu) = a^\mu b^\nu \mathrm{Tr}(\gamma_\mu\gamma_\nu) = a^\mu b^\nu \mathrm{Tr}(2g_{\mu\nu}\mathbf{1}_{4\times4} - \gamma_\nu\gamma_\mu) \tag{11}$$

$$= 8a^\mu b^\nu g_{\mu\nu} - a^\mu b^\nu \mathrm{Tr}(\gamma_\nu\gamma_\mu) \tag{12}$$

Note for $\mu \neq \nu$,

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = -\mathrm{Tr}(\gamma^\nu\gamma^\mu) = -\mathrm{Tr}(\gamma^\mu\gamma^\nu) \tag{13}$$

Eq.(10) then implies the following

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = g^{\mu\nu}\mathrm{Tr}(\mathbf{1}_{4\times4}) \tag{14}$$

Which implies that $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 0$ for $\mu \neq \nu$. This leaves us with

$$= 8a^\mu b^\nu g_{\mu\nu} - a^\mu b^\nu g_{\mu\nu}\mathrm{Tr}(\mathbf{1}_{4\times4}) \tag{15}$$

$$= 8a^\mu b_\mu - 4a^\mu b_\mu = 4a^\mu b_\mu \tag{16}$$

1.3 $\mathbf{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4\left[(a\cdot b)(c\cdot d) + (a\cdot d)(b\cdot c) - (a\cdot c)(b\cdot d)\right]$

$$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = a^\mu b^\nu c^\sigma d^\lambda \mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\lambda) \tag{17}$$

$$= a^\mu b^\nu c^\sigma d^\lambda \mathrm{Tr}((2g_{\mu\nu}\mathbf{1}_{4\times4} - \gamma_\nu\gamma_\mu)\gamma_\sigma\gamma_\lambda) \tag{18}$$

$$= a^\mu b^\nu c^\sigma d^\lambda \left(2g_{\mu\nu}\mathrm{Tr}(\gamma_\sigma\gamma_\lambda) - \mathrm{Tr}(\gamma_\nu\gamma_\mu\gamma_\sigma\gamma_\lambda)\right) \tag{19}$$

$$= a^\mu b^\nu c^\sigma d^\lambda (8g_{\mu\nu}g_{\sigma\lambda} - \mathrm{Tr}(\gamma_\nu(2g_{\mu\sigma}\mathbf{1}_{4\times4} - \gamma_\sigma\gamma_\mu)\gamma_\lambda)) \tag{20}$$

$$= a^\mu b^\nu c^\sigma d^\lambda (8g_{\mu\nu}g_{\sigma\lambda} - 8g_{\mu\sigma}g_{\nu\lambda} + \mathrm{Tr}(\gamma_\nu\gamma_\sigma\gamma_\mu\gamma_\lambda)) \tag{21}$$

$$= 8a^\mu b_\mu c^\sigma d_\sigma - 8a^\mu c_\mu b^\nu d_\nu + a^\mu b^\nu c^\sigma d^\lambda \mathrm{Tr}(\gamma_\nu\gamma_\sigma\gamma_\mu\gamma_\lambda) \tag{22}$$

$$= 8a^\mu b_\mu c^\sigma d_\sigma - 8a^\mu c_\mu b^\nu d_\nu + a^\mu b^\nu c^\sigma d^\lambda \big(\mathrm{Tr}[\gamma_\nu\gamma_\sigma(2g_{\mu\lambda}\mathbf{1}_{4\times4} - \gamma_\lambda\gamma_\mu)]\big) \tag{23}$$

$$= 8a^\mu b_\mu c^\sigma d_\sigma - 8a^\mu c_\mu b^\nu d_\nu + a^\mu b^\nu c^\sigma d^\lambda \big(8g_{\nu\sigma}g_{\mu\lambda} - \mathrm{Tr}[\gamma_\nu\gamma_\sigma\gamma_\lambda\gamma_\mu]\big) \tag{24}$$

$$= 8a^\mu b_\mu c^\sigma d_\sigma - 8a^\mu c_\mu b^\nu d_\nu + 8a^\mu d_\mu b^\nu c_\nu - \mathrm{Tr}(\gamma_\nu\gamma_\sigma\gamma_\lambda\gamma_\mu) \tag{25}$$

$$= 8a^\mu b_\mu c^\sigma d_\sigma - 8a^\mu c_\mu b^\nu d_\nu + 8a^\mu d_\mu b^\nu c_\nu - \mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\lambda) \tag{26}$$

Let's rewrite for explicitness

$$a^\mu b^\nu c^\sigma d^\lambda \mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\lambda) = 8a^\mu b_\mu c^\sigma d_\sigma - 8a^\mu c_\mu b^\nu d_\nu + 8a^\mu d_\mu b^\nu c_\nu - a^\mu b^\nu c^\sigma d^\lambda \mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\lambda)$$

$$2a^\mu b^\nu c^\sigma d^\lambda \mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\lambda) = 8a^\mu b_\mu c^\sigma d_\sigma - 8a^\mu c_\mu b^\nu d_\nu + 8a^\mu d_\mu b^\nu c_\nu \tag{27}$$

$$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4[a^\mu b_\mu c^\sigma d_\sigma - a^\mu c_\mu b^\nu d_\nu + a^\mu d_\mu b^\nu c_\nu] \tag{28}$$

$$= 4\left[(a\cdot b)(c\cdot d) + (a\cdot d)(b\cdot c) - (a\cdot c)(b\cdot d)\right] \tag{29}$$

1.4 $\gamma_\mu \not{a} \gamma^\mu = -2\not{a}$

$$\gamma_\mu \not{a} \gamma^\mu = \gamma_\mu a_\nu \gamma^\nu \gamma^\mu \tag{30}$$

$$= \gamma_\mu (2g^{\nu\mu} \mathbb{1}_{4\times 4} - \gamma^\mu \gamma^\nu) a_\nu \tag{31}$$

$$= 2\not{a} + \gamma_\mu \gamma^\mu \not{a} \tag{32}$$

Note the following

$$\gamma_\mu \gamma^\mu = g_{\mu\nu} \gamma^\nu \gamma^\mu = \frac{1}{2}(g_{\mu\nu} + g_{\nu\mu}) \gamma^\nu \gamma^\mu \tag{33}$$

Where I have decomposed $g^{\mu\nu}$ into its symmetric and antisymmetric parts[1] (antisymmetric part is zero)

$$= \frac{1}{2}(g_{\mu\nu} \gamma^\nu \gamma^\mu + g_{\nu\mu} \gamma^\nu \gamma^\mu) \tag{34}$$

Relabeling indices on the second term

$$= \frac{1}{2}(g_{\mu\nu} \gamma^\nu \gamma^\mu + g_{\mu\nu} \gamma^\mu \gamma^\nu) \tag{35}$$

$$= \frac{1}{2} g_{\mu\nu} \{\gamma^\mu, \gamma^\nu\} = g_{\mu\nu} g^{\mu\nu} \mathbb{1}_{4\times 4} = 4\mathbb{1}_{4\times 4} \tag{36}$$

Plugging this result back into Eq.(32) yields

$$= 2\not{a} - 4\not{a} = -2\not{a} \tag{37}$$

1.5 $\gamma_\mu \not{a} \not{b} \gamma^\mu = 4a \cdot b$

$$\gamma_\mu \not{a} \not{b} \gamma^\mu = \gamma_\mu \gamma^\nu \gamma^\sigma \gamma^\mu a_\nu b_\sigma \tag{38}$$

$$= \gamma_\mu \gamma^\nu (2g^{\sigma\mu} \mathbb{1}_{4\times 4} - \gamma^\mu \gamma^\sigma) a_\nu b_\sigma \tag{39}$$

$$= 2\not{b}\not{a} - \gamma_\mu (2g^{\nu\mu} \mathbb{1}_{4\times 4} - \gamma^\mu \gamma^\nu) a_\nu \not{b} \tag{40}$$

$$= 2\not{b}\not{a} - 2\not{a}\not{b} + \gamma_\mu \gamma^\mu \not{a}\not{b} \tag{41}$$

$$= 4(a \cdot b) - 4\not{a}\not{b} + 4\not{a}\not{b} \tag{42}$$

$$= 4(a \cdot b) \tag{43}$$

[1] Generally any tensor A_{ij} of rank two can be decomposed into a sum of a symmetric and antisymmetric tensor i.e. $A_{ij} = B_{ij} + C_{ij} = \frac{1}{2}(A_{ij} + A_{ji}) + \frac{1}{2}(A_{ij} - A_{ji})$ where it can be seen in the second equality that $B_{ij} \equiv \frac{1}{2}(A_{ij} + A_{ji})$ and $C_{ij} \equiv \frac{1}{2}(A_{ij} - A_{ji})$

1.6 $\gamma_\mu \not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a}$

$$\gamma_\mu \not{a}\not{b}\not{c}\gamma^\mu = \gamma_\mu \gamma^\nu \gamma^\sigma \gamma^\lambda \gamma^\mu a_\nu b_\sigma c_\lambda \tag{44}$$
$$= \gamma_\mu \gamma^\nu \gamma^\sigma (2g^{\lambda\mu} - \gamma^\mu \gamma^\lambda) a_\nu b_\sigma c_\lambda \tag{45}$$
$$= 2\not{c}\not{a}\not{b} - \gamma_\mu \gamma^\nu (2g^{\sigma\mu} - \gamma^\mu \gamma^\sigma) a_\nu b_\sigma \not{c} \tag{46}$$
$$= 2\not{c}\not{a}\not{b} - 2\not{b}\not{a}\not{c} + \gamma_\mu (2g^{\nu\mu} - \gamma^\mu \gamma^\nu) a_\nu \not{b}\not{c} \tag{47}$$
$$= 2\not{c}\not{a}\not{b} - 2\not{b}\not{a}\not{c} + 2\not{a}\not{b}\not{c} - \gamma_\mu \gamma^\mu \not{a}\not{b}\not{c} \tag{48}$$
$$= 2\not{c}\not{a}\not{b} - 2\not{b}\not{a}\not{c} - 2\not{a}\not{b}\not{c} \tag{49}$$
$$= 2\not{c}\not{a}\not{b} - 2\left(\not{b}\not{a} + \not{a}\not{b}\right)\not{c} = 2\not{c}\not{a}\not{b} - 2\{\gamma^\sigma, \gamma^\nu\} a_\nu b_\sigma \not{c} \tag{50}$$
$$= 2\not{c}\not{a}\not{b} - 4(a \cdot b)\not{c} \tag{51}$$
$$= 2\not{c}(2(a \cdot b) - \not{b}\not{a}) - 4(a \cdot b)\not{c} \tag{52}$$
$$= -2\not{c}\not{b}\not{a} \tag{53}$$

2 Dirac algebra

Show that the Dirac matrices defined in the lecture satisfy the identities using the Clifford relation rather than an explicit representation.

The Clifford relation is given by

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}_{4\times 4} \tag{54}$$

2.1 $\mathbf{Tr}(\prod_{i=1}^{\mathbf{odd}} \gamma^{\mu_i}) = 0$

First note the following property of the gamma martices

$$\mathrm{Tr}(\gamma^\mu) = 0 \tag{55}$$

The first non-trivial case is the prodcut of three gamma matrices

$$\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}) \tag{56}$$
$$= \mathrm{Tr}(2g^{\mu_1\mu_2}\mathbf{1}_{4\times 4}\gamma^{\mu_3} - \gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_3}) \tag{57}$$
$$= 2g^{\mu_1\mu_2}\mathrm{Tr}(\gamma^{\mu_3}) - \mathrm{Tr}(\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_3}) \tag{58}$$
$$= -\mathrm{Tr}(\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_3}) = -\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}) \tag{59}$$

Which implies $\text{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}) = 0$.

Generalizing to the product of n gamma matrices with n being some odd number

$$\text{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\cdots\gamma^{\mu_n}) \tag{60}$$

We can follow the same procedure

$$= 2g^{\mu_1\mu_2}\text{Tr}(\gamma^{\mu_3}\gamma^{\mu_4}\cdots\gamma^{\mu_n}) - \text{Tr}(\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_3}\cdots\gamma^{\mu_n}) \tag{61}$$

$$= 2g^{\mu_1\mu_2}\left[2g^{\mu_3\mu_4}\text{Tr}(\gamma^{\mu_5}\gamma^{\mu_6}\cdots\gamma^{\mu_n}) - \text{Tr}(\gamma^{\mu_4}\gamma^{\mu_3}\gamma^{\mu_5}\cdots\gamma^{\mu_n})\right] - \text{Tr}(\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_3}\cdots\gamma^{\mu_n}) \tag{62}$$

We can continue this process for each of the traces within the square brackets but because n is odd in each of these terms we can continue to do this until we are left a sum of $\text{Tr}(\gamma^{\mu_n})$. Thus, we are just left with

$$= -\text{Tr}(\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_3}\cdots\gamma^{\mu_n}) = -\text{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\cdots\gamma^{\mu_n}) \tag{63}$$

Implying that $\text{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\cdots\gamma^{\mu_n}) = 0$

A slicker proof utilizes $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$

2.2 $\{\gamma^5, \gamma^\mu\} = 0$

First noting

$$(\gamma^0)^2 = \mathbf{1}_{4\times4}, \qquad (\gamma^k)^2 = -\mathbf{1}_{4\times4} \qquad \text{for } k = 1,2,3 \tag{64}$$

$$\{\gamma^5, \gamma^\mu\} = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 \tag{65}$$

$$= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + i(2g^{\mu0}\gamma^1\gamma^2\gamma^3 - \gamma^0\gamma^\mu\gamma^1\gamma^2\gamma^3) \tag{66}$$

$$= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + 2ig^{\mu0}\gamma^1\gamma^2\gamma^3 - i(2g^{\mu1}\gamma^0\gamma^2\gamma^3 - \gamma^0\gamma^1\gamma^\mu\gamma^2\gamma^3) \tag{67}$$

$$= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + 2ig^{\mu0}\gamma^1\gamma^2\gamma^3 - 2ig^{\mu1}\gamma^0\gamma^2\gamma^3 + i(2g^{\mu2}\gamma^0\gamma^1\gamma^3 - \gamma^0\gamma^1\gamma^2\gamma^\mu\gamma^3) \tag{68}$$

$$= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + 2ig^{\mu0}\gamma^1\gamma^2\gamma^3 - 2ig^{\mu1}\gamma^0\gamma^2\gamma^3 + 2ig^{\mu2}\gamma^0\gamma^1\gamma^3 - i(2g^{\mu3}\gamma^0\gamma^1\gamma^2 - \gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu)$$

$$= 2i(g^{\mu0}\gamma^1\gamma^2\gamma^3 - g^{\mu1}\gamma^0\gamma^2\gamma^3 + g^{\mu2}\gamma^0\gamma^1\gamma^3 - g^{\mu3}\gamma^0\gamma^1\gamma^2 + \gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu) \tag{69}$$

Now I can check for each value of μ

1. $\mu = 0$

$$2i(\gamma^1\gamma^2\gamma^3 + \gamma^0\gamma^1\gamma^2\gamma^3\gamma^0) \tag{70}$$

$$= 2i(\gamma^1\gamma^2\gamma^3 + (-1)^3\gamma^1\gamma^2\gamma^3\gamma^0\gamma^0) \tag{71}$$

$$= 0 \tag{72}$$

2. $\mu = 1$

$$2i(-\gamma^0\gamma^2\gamma^3 + \gamma^0\gamma^1\gamma^2\gamma^3\gamma^1) \tag{73}$$
$$= 2i(\gamma^0\gamma^2\gamma^3 + (-1)^2\gamma^0\gamma^2\gamma^3\gamma^1\gamma^1) \tag{74}$$
$$= 0 \tag{75}$$

3. $\mu = 2$

$$2i(-\gamma^0\gamma^1\gamma^3 + \gamma^0\gamma^1\gamma^2\gamma^3\gamma^2) \tag{76}$$
$$= 2i(-\gamma^0\gamma^1\gamma^3 + (-1)^1\gamma^0\gamma^1\gamma^3\gamma^2\gamma^2) \tag{77}$$
$$= 0 \tag{78}$$

4. $\mu = 3$

$$2i(\gamma^0\gamma^1\gamma^2 + \gamma^0\gamma^1\gamma^2\gamma^3\gamma^3) \tag{79}$$
$$= 0 \tag{80}$$

2.3 Tr(γ^5) =0

$$\mathrm{Tr}(\gamma^5) = \mathrm{Tr}(i\gamma^0\gamma^1\gamma^2\gamma^3) = (-1)^3 i\mathrm{Tr}(\gamma^1\gamma^2\gamma^3\gamma^0) = -i\mathrm{Tr}(\gamma^0\gamma^1\gamma^2\gamma^3) = -\mathrm{Tr}(\gamma^5) \tag{81}$$

Thus, $\mathrm{Tr}(\gamma^5) = 0$

2.4 $(\gamma^5)^2 = \mathbf{1}_{4\times 4}$

$$(\gamma^5)^2 = \gamma^5\gamma^5 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = -(-1)^3\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 = (-1)^2\gamma^1\gamma^1\gamma^2\gamma^3\gamma^2\gamma^3$$
$$= -(-1)^1\gamma^2\gamma^2\gamma^3\gamma^3 = \mathbf{1}_{4\times 4} \tag{82}$$

Now using (when neccesary) the chiral representation given by,

$$\gamma^0 = \begin{pmatrix} 0 & \underline{\mathbb{1}} \\ \underline{\mathbb{1}} & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \underline{\sigma}^i \\ -\underline{\sigma}^i & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} -\underline{\mathbb{1}} & 0 \\ 0 & \underline{\mathbb{1}} \end{pmatrix} \qquad i = 1, 2, 3 \tag{83}$$

where the underline indicates a two by two matrix. Show that

2.5 $\beta\gamma^{\mu\dagger}\beta = \gamma^\mu$

Where $\beta \equiv \gamma^0$

1. $\mu = 0$
$$\gamma^0\gamma^{0\dagger}\gamma^0 = \gamma^0\gamma^0\gamma^0 = \gamma^0 \tag{84}$$

2. $\mu = 1$
$$\gamma^0\gamma^{1\dagger}\gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{85}$$
$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{86}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \gamma^1 \tag{87}$$

3. $\mu = 2$
$$\gamma^0\gamma^{1\dagger}\gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{88}$$
$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \tag{89}$$
$$= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = \gamma^2 \tag{90}$$

4. $\mu = 3$

$$\gamma^0 \gamma^{1\dagger} \gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{91}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{92}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \gamma^3 \tag{93}$$

2.6 $\gamma^5 = \gamma^{5\dagger}$

Eq.(83) shows γ^5 is diagonal and has real entries thus,

$$\gamma^{5\dagger} = \gamma^5 \tag{94}$$

Also from Eq.(83) we see that

$$\gamma^{\mu\dagger} = g^{[\mu\mu]} \gamma^\mu \tag{95}$$

Where no summation is implied, $g^{[\mu\mu]}$ is a place holder for positive and negative values i.e. $g^{[11]} = 1, g^{[22]} = -1, g^{[33]} = -1, g^{[44]} = -1$ implying

$$(\gamma^0)^\dagger = \gamma^0, \qquad (\gamma^i)^\dagger = -\gamma^i \qquad \text{for } i = 1, 2, 3 \tag{96}$$

I could also write this as

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0 \tag{97}$$

Noting that $g^{[\mu\mu]} g^{[\nu\nu]} = 1$

$$\mathcal{J}^{\mu\nu\dagger} = \frac{i}{4}\{\gamma^\mu, \gamma^\nu\}^\dagger = \frac{i}{4}\{\gamma^{\mu\dagger}, \gamma^{\nu\dagger}\} = \frac{i}{4} g^{[\mu\mu]} g^{[\nu\nu]} \{\gamma^\mu, \gamma^\nu\} = \mathcal{J}^{\mu\nu} \tag{98}$$

$$(\gamma^\mu \gamma^5)^\dagger = \gamma^{5\dagger} \gamma^{\mu\dagger} = -g^{[\mu\mu]} \gamma^5 \gamma^\mu = g^{[\mu\mu]} \gamma^\mu \gamma^5 \tag{99}$$

2.7 $(\gamma^\mu)^T = -\mathcal{C}\gamma^\mu\mathcal{C}^{-1}$

Where $\mathcal{C} \equiv -i\gamma^2\beta$

$$\mathcal{C} = -i\begin{pmatrix} 0 & \underline{\sigma}^2 \\ -\underline{\sigma}^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \underline{\mathbb{1}} \\ \underline{\mathbb{1}} & 0 \end{pmatrix} = -i\begin{pmatrix} \underline{\sigma}^2 & 0 \\ 0 & -\underline{\sigma}^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \underline{\sigma}_1 & 0 \\ 0 & -\underline{\sigma}_1 \end{pmatrix} \tag{100}$$

$$\mathcal{C}^{-1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{101}$$

1. $\mu = 0$

$$\begin{aligned} -\mathcal{C}\gamma^0\mathcal{C}^{-1} &= -\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \gamma^0 = (\gamma^0)^T \end{aligned} \tag{102}$$

2. $\mu = 1$

$$\begin{aligned} -\mathcal{C}\gamma^1\mathcal{C}^{-1} &= -\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = (\gamma^1)^T \end{aligned} \tag{103}$$

3. $\mu = 2$

$$-\mathcal{C}\gamma^2\mathcal{C}^{-1} = -\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = \gamma^2 = (\gamma^2)^T \tag{104}$$

4. $\mu = 3$

$$-\mathcal{C}\gamma^3\mathcal{C}^{-1} = -\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = (\gamma^3)^T \tag{105}$$

Likewise,

$$(\gamma^5)^T = \gamma^5 \tag{106}$$

$$(\gamma^\mu\gamma^5)^T = (\gamma^5)^T(\gamma^\mu)^T = -\gamma^5\mathcal{C}\gamma^\mu\mathcal{C}^{-1} = \mathcal{C}\gamma^\mu\gamma^5\mathcal{C}^{-1} \tag{107}$$

Where I have used the fact that $[\gamma^5, \mathcal{C}] = 0$ and $\{\gamma^5, \gamma^\mu\} = 0$.

$$(\mathcal{J}^{\mu\nu})^T = \mathcal{J}^{\nu\mu} = -\mathcal{J}^{\mu\nu} \tag{108}$$

Due to the antisymmetry of $\mathcal{J}^{\mu\nu}$.

Under complex conjugation we have

$$\gamma^{5*} = \gamma^5 \tag{109}$$

$$(\gamma^\mu\gamma^5)^* = ((\gamma^\mu\gamma^5)^\dagger)^T = (g^{[\mu\mu]}\gamma^\mu\gamma^5)^T = g^{[\mu\mu]}\mathcal{C}\gamma^\mu\gamma^5\mathcal{C}^{-1} \tag{110}$$

$$(\mathcal{J}^{\mu\nu})^* = ((\mathcal{J}^{\mu\nu})^\dagger)^T = (\mathcal{J}^{\mu\nu})^T = -\mathcal{J}^{\mu\nu} \tag{111}$$

$$\gamma^{\mu*} = ((\gamma^\mu)^\dagger)^T = (g^{[\mu\mu]}\gamma^\mu)^T = -g^{[\mu\mu]}\mathcal{C}\gamma^\mu\mathcal{C}^{-1} = g_{[\mu\mu]}\mathcal{C}\gamma^\mu\mathcal{C}^{-1} \tag{112}$$

Where I have abused notation for convenience, $g_{[00]} = -1, g_{[ii]} = 1$ for $i = 1, 2, 3$.

2.8 $\gamma^\mu\gamma^\nu\gamma^\rho = g^{\mu\nu}\gamma^\rho + g^{\nu\rho}\gamma^\mu - g^{\mu\rho}\gamma^\nu - i\varepsilon^{\lambda\mu\nu\rho}\gamma_\lambda\gamma^5$

There are $4^3 = 64$ combinations of indices, we consider

1. $\mu = \nu = \rho$ $[\times 4]$

$$\gamma^0\gamma^0\gamma^0 = g^{00}\gamma^0 + g^{00}\gamma^0 - g^{00}\gamma^0 = \gamma^0 \tag{113}$$

$$\gamma^i\gamma^i\gamma^i = g^{ii}\gamma^i + g^{ii}\gamma^i - g^{ii}\gamma^i = -\gamma^i \tag{114}$$

2. $\mu = \nu \neq \rho, \mu \neq \nu = \rho, \mu = \rho \neq \nu$ $[\times 36]$

All of these cases are very similar, I'll illustrate the general idea for $\mu = \nu \neq \rho$. On the left hand side we obtain

$$\gamma^\mu\gamma^\mu\gamma^\rho = g^{[\mu\mu]}\gamma^\rho \tag{115}$$

On the right hand side

$$g^{[\mu\mu]}\gamma^\rho + g^{\mu\rho}\gamma^\mu g^{\mu\rho}\gamma^\nu - i\varepsilon^{\lambda\mu\mu\rho}\gamma_\lambda\gamma^5 = g^{[\mu\mu]}\gamma^\rho \tag{116}$$

3. $\mu \neq \nu \neq \rho$ $[\times 24]$

Again, all of the combinations are very similar, I will do a few examples to illustrate. In this case all that is left on the right hand side is the anti-symmetric symbol

(a) $\mu = 0, \nu = 1, \rho = 2$

$$\gamma^0\gamma^1\gamma^2 = -i\varepsilon^{3012}\gamma_3\gamma^5 = \varepsilon^{3012}(-\gamma^3)\gamma^0\gamma^1\gamma^2\gamma^3 = (-1)(-1)(-1)^3\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3 = \gamma^0\gamma^1\gamma^2 \tag{117}$$

(b) $\mu = 1, \nu = 3, \rho = 0$

$$\gamma^1\gamma^3\gamma^0 = -i\varepsilon^{2130}\gamma_2\gamma^5 = \varepsilon^{2130}(-\gamma^2)\gamma^0\gamma^1\gamma^2\gamma^3 = -\varepsilon^{2130}(-1)^2\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3 \tag{118}$$

$$= \gamma^0\gamma^1\gamma^3 = (-1)^2\gamma^1\gamma^3\gamma^0 = \gamma^1\gamma^3\gamma^0 \tag{119}$$

(c) One last example: $\mu = 3, \nu = 2, \rho = 1$

$$\gamma^3\gamma^2\gamma^1 = -i\varepsilon^{0321}\gamma_0\gamma^5 = \varepsilon^{0321}\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^1\gamma^2\gamma^3 = -(-1)^3\gamma^3\gamma^2\gamma^1 \tag{120}$$

$$= \gamma^3\gamma^2\gamma^1 \tag{121}$$

The other permutations follow very closely to those above.

3 More Dirac Algebra

Show the following relations

3.1 $\mathrm{Tr}[\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\gamma^5] = -4i\varepsilon^{\mu_1\mu_1\mu_3\mu_4}$

The only contributing case is when $\mu_1 \neq \mu_2 \neq \mu_3 \neq \mu_4$. In this case, the trace is proportional to the antisymmetric symbol due to the anticommutation of gamma matrices with differing indices. To find the proportionality constant we can simply do a test case of $\mu_1\mu_2\mu_3\mu_4 = 0123$.

$$\mathrm{Tr}[\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5] = i\mathrm{Tr}[\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3] \tag{122}$$

$$= i(-1)^3\mathrm{Tr}[\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3] \tag{123}$$

$$= -i(-1)^2\mathrm{Tr}[\gamma^1\gamma^1\gamma^2\gamma^3\gamma^2\gamma^3] \tag{124}$$

$$= i(-1)\mathrm{Tr}[\gamma^2\gamma^2\gamma^3\gamma^3] \tag{125}$$

$$= -i\mathrm{Tr}[\mathbb{1}_{4\times 4}] = -4i \tag{126}$$

Thus, the proportionality constant is $-4i$ and

$$\mathrm{Tr}[\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\gamma^5] = -4i\varepsilon^{\mu_1\mu_1\mu_3\mu_4} \tag{127}$$

3.2 $\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$

$$\gamma^\mu\gamma^\nu\gamma_\mu = 2g^{\mu\nu}\gamma_\mu - \gamma^\nu\gamma^\mu\gamma_\mu \tag{128}$$

Noting

$$\gamma^\mu\gamma_\mu = \gamma_\mu\gamma^\mu = g_{\mu\nu}\gamma^\nu\gamma^\mu = \frac{1}{2}(g_{\mu\nu} + g_{\nu\mu})\gamma^\nu\gamma^\mu \tag{129}$$

Where I have decomposed $g^{\mu\nu}$ into its symmetric and antisymmetric parts[2] (antisymmetric part is zero)

$$= \frac{1}{2}(g_{\mu\nu}\gamma^\nu\gamma^\mu + g_{\nu\mu}\gamma^\nu\gamma^\mu) \tag{130}$$

Relabeling indices on the second term

$$= \frac{1}{2}(g_{\mu\nu}\gamma^\nu\gamma^\mu + g_{\mu\nu}\gamma^\mu\gamma^\nu) \tag{131}$$

[2]Generally any tensor A_{ij} of rank two can be decomposed into a sum of a symmetric and antisymmetric tensor i.e. $A_{ij} = B_{ij} + C_{ij} = \frac{1}{2}(A_{ij} + A_{ji}) + \frac{1}{2}(A_{ij} - A_{ji})$ where it can be seen in the second equality that $B_{ij} \equiv \frac{1}{2}(A_{ij} + A_{ji})$ and $C_{ij} \equiv \frac{1}{2}(A_{ij} - A_{ji})$

$$= \frac{1}{2} g_{\mu\nu}\{\gamma^\mu, \gamma^\nu\} = g_{\mu\nu} g^{\mu\nu} \mathbb{1}_{4\times 4} = 4\mathbb{1}_{4\times 4} \tag{132}$$

Plugging this result back into Eq.(128)

$$= 2g^{\mu\nu}\gamma_\mu - 4\gamma^\nu \tag{133}$$

$$= -2\gamma^\nu \tag{134}$$

3.3 $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 2g^{\mu\nu}\gamma^\rho\gamma_\mu - \gamma^\nu\gamma^\mu\gamma^\rho\gamma_\mu \tag{135}$$

$$= 2g^{\mu\nu}\gamma^\rho\gamma_\mu - 2g^{\mu\rho}\gamma^\nu\gamma_\mu + \gamma^\nu\gamma^\rho\gamma^\mu\gamma_\mu \tag{136}$$

$$= 2g^{\mu\nu}\gamma^\rho\gamma_\mu - 2g^{\mu\rho}\gamma^\nu\gamma_\mu + 4\gamma^\nu\gamma^\rho \tag{137}$$

$$= 2\gamma^\rho\gamma^\mu - 2\gamma^\nu\gamma^\rho + 4\gamma^\nu\gamma^\rho = 2(\gamma^\rho\gamma^\nu + \gamma^\nu\gamma^\rho) \tag{138}$$

$$= 2\{\gamma^\nu, \gamma^\rho\} = 4g^{\nu\rho} \tag{139}$$

3.4 $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = 2g^{\mu\nu}\gamma^\rho\gamma^\sigma\gamma_\mu - \gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma\gamma_\mu \tag{140}$$

$$= 2g^{\mu\nu}\gamma^\rho\gamma^\sigma\gamma_\mu - g^{\mu\rho}\gamma^\nu\gamma^\sigma\gamma_\mu + \gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma\gamma_\mu \tag{141}$$

$$= 2g^{\mu\nu}\gamma^\rho\gamma^\sigma\gamma_\mu - 2g^{\mu\rho}\gamma^\nu\gamma^\sigma\gamma_\mu + 2g^{\mu\sigma}\gamma^\nu\gamma^\rho\gamma_\mu - \gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu\gamma_\mu \tag{142}$$

$$= 2\gamma^\rho\gamma^\sigma\gamma^\nu - 2\gamma^\nu\gamma^\sigma\gamma^\rho + 2\gamma^\nu\gamma^\rho\gamma^\sigma - 4\gamma^\nu\gamma^\rho\gamma^\sigma \tag{143}$$

$$= 2\gamma^\rho\gamma^\sigma\gamma^\nu - 2\gamma^\nu\gamma^\sigma\gamma^\rho - 2\gamma^\nu\gamma^\rho\gamma^\sigma \tag{144}$$

$$= 2\gamma^\rho\gamma^\sigma\gamma^\nu - 2\gamma^\nu(\gamma^\sigma\gamma^\rho + \gamma^\rho\gamma^\sigma) = 2\gamma^\rho\gamma^\sigma\gamma^\nu - 2\gamma^\nu\{\gamma^\sigma, \gamma^\rho\} \tag{145}$$

$$= 4g^{\rho\sigma}\gamma^\nu - 2\gamma^\sigma\gamma^\rho\gamma^\nu - 4g^{\rho\sigma}\gamma^\nu \tag{146}$$

$$= -2\gamma^\sigma\gamma^\rho\gamma^\nu \tag{147}$$