COMPTON SCATTERING

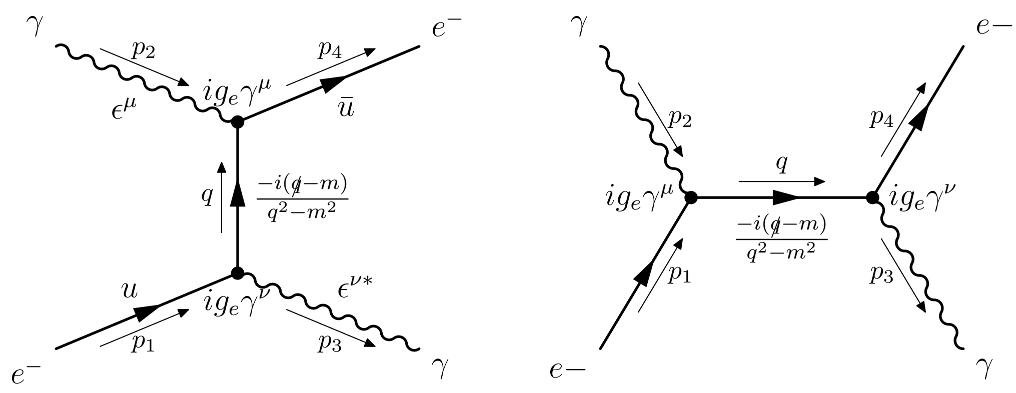
November 13, 2020

Tony Menzo

1 Compton scattering

Here we compute the differential cross section for Compton scattering ($e^-\gamma \to e^-\gamma$) as a function of the polar angle θ, evaluated in the rest system of the initial-state electron, between the direction of the incoming photon and the out-going photon.

1.1 Diagrams



[A] [B]

Figure 1: [A] Shows the t-channel Feynman diagram and [B] shows the s-channel Feynman diagram

1.2 Matrix Elements

The scattering amplitude for the t-channel diagram is given by

$$-2\pi i\delta^4(p_3+p_4-p_2-p_1)\mathcal{M}_t = \int \frac{\epsilon_\mu(\mathbf{p}_2,\sigma_2)}{(2\pi)^{3/2}\sqrt{2p_2^0}}\left[\frac{\bar u(\mathbf{p}_4,\sigma_4)}{(2\pi)^{3/2}}(-ie)\gamma^\mu \frac{1}{(2\pi)^4}\frac{i(\not q + m_e)}{q^2-m_e^2+i\varepsilon}(-ie)\gamma^\nu\frac{u(\mathbf{p}_1,\sigma_1)}{(2\pi)^{3/2}}\right]\frac{\epsilon^*_\nu(\mathbf{p}_3,\sigma_3)}{(2\pi)^{3/2}\sqrt{2p_3^0}} \times (2\pi)^4\delta^4(p_2+q-p_4)(2\pi)^4\delta^4(p_1-q-p_3)d^4q \tag{1}$$

Switching to a more concise, but still clear, notation

$$= \frac{-ie^2}{(2\pi)^2(q'^2-m_e^2)\sqrt{4p_2^0p_3^0}}[\bar u_4 \not\epsilon_2(\not q' + m_e)\not\epsilon_3^* u_1]\delta^4(p_4+p_3-p_2-p_1) \tag{2}$$

Where $q' \equiv p_1 - p_3$. Thus,

$$\mathcal{M}_t = \frac{-e^2}{(2\pi)^3(q'^2 - m_e^2)\sqrt{4p_2^0 p_3^0}}[\bar{u}_4 \not{\epsilon}_2 (\not{q}' + m_e)\not{\epsilon}_3^* u_1] \tag{3}$$

This can be simplified further by noting the following

$$(\not{p}_1 - \not{p}_3 + m_e)\gamma^\nu u_1 = -\not{p}_3 \gamma^\nu u_1 + (\gamma^\rho \gamma^\nu p_{1\rho} + \gamma^\nu m_e)u_1 \tag{4}$$

$$= -\not{p}_3 \gamma^\nu u_1 + ([2g^{\rho\nu} - \gamma^\nu \gamma^\rho]p_{1\rho} + \gamma^\nu m_e)u = -\not{p}_3 \gamma^\nu u_1 + 2p^\nu u - \gamma^\nu(\not{p}_1 - m_e)u_1 \tag{5}$$

Using the Dirac equation $(\not{p}_1 - m_e)u_1$

$$= (-\not{p}_3 \gamma^\nu + 2p^\nu)u_1 \tag{6}$$

The denominator can also be simplified

$$q'^2 - m_e^2 = p_1^2 - p_3^2 - 2(p_1 \cdot p_3) - m_e^2 = -2(p_1 \cdot p_3) \tag{7}$$

Where I have used the fact that $p_1^2 = m_e^2$, $p_3^2 = 0$.

The matrix element becomes

$$\mathcal{M}_t = \frac{e^2}{(2\pi)^3 2(p_1 \cdot p_3)\sqrt{4p_2^0 p_3^0}}[\bar{u}_4 \not{\epsilon}_2 (2p_1^\nu - \not{p}_3 \gamma^\nu)\epsilon_{3\nu}^* u_1] \tag{8}$$

$$= \frac{e^2}{(2\pi)^3 2(p_1 \cdot p_3)\sqrt{4p_2^0 p_3^0}}[\bar{u}_4 \epsilon_{2\mu}(\gamma^\mu 2p_1^\nu - \gamma^\mu \not{p}_3 \gamma^\nu)\epsilon_{3\nu}^* u_1] \tag{9}$$

For the s-channel diagram we have,

$$-2\pi i\delta^4(p_3 + p_4 - p_2 - p_1)\mathcal{M}_s =$$
$$\int \frac{\epsilon_\mu^*(\mathbf{p}_3, \sigma_3)}{(2\pi)^{3/2}\sqrt{2p_3^0}} \left[\frac{\bar{u}(\mathbf{p}_4, \sigma_4)}{(2\pi)^{3/2}}(-ie)\gamma^\mu \frac{1}{(2\pi)^4}\frac{i(\not{q} + m_e)}{q^2 - m_e^2 + i\varepsilon}(-ie)\gamma^\nu \frac{u(\mathbf{p}_1, \sigma_1)}{(2\pi)^{3/2}}\right] \frac{\epsilon_\nu(\mathbf{p}_2, \sigma_2)}{(2\pi)^{3/2}\sqrt{2p_2^0}}$$
$$\times (2\pi)^4 \delta^4(p_1 + p_2 - q(2\pi)^4 \delta^4(q - p_3 - p_4)d^4q \tag{10}$$

Again switching up notation,

$$= \frac{-ie^2}{(2\pi)^2(q''^2 - m_e^2)\sqrt{4p_2^0 p_3^0}}[\bar{u}_4 \not{\epsilon}_3^* (\not{q}'' + m_e)\not{\epsilon}_2 u_1]\delta^4(p_4 + p_3 - p_2 - p_1) \tag{11}$$

Where $q'' \equiv p_1 + p_2$. Thus,

$$\mathcal{M}_s = \frac{-e^2}{(2\pi)^3(q''^2 - m_e^2)\sqrt{4p_2^0 p_3^0}}[\bar{u}_4 \not{\epsilon}_3^* (\not{q}'' + m_e)\not{\epsilon}_2 u_1] \tag{12}$$

This can be simplified as well

$$(\not{p}_2\gamma^\nu + \gamma^\sigma\gamma^\nu p_{1\sigma} + m_e\gamma^\mu)u_1 = \not{p}_2\gamma^\nu u_1 + [(2g^{\sigma\nu}p_{1\sigma} - \gamma^\nu\not{p}_1) + m_e\gamma^\nu]u_1 \tag{13}$$

$$= (2p^\nu + \not{p}_2\gamma^\nu)u_1 \tag{14}$$

$$\mathcal{M}_s = \frac{-e^2}{(2\pi)^3 2(p_1\cdot p_2)\sqrt{4p_2^0 p_3^0}}[\bar{u}_4 \epsilon_{3\mu}^* (2\gamma^\mu p^\nu + \gamma^\mu\not{p}_2\gamma^\nu)\epsilon_{2\nu}u_1] \tag{15}$$

The total amplitude is then given by

$$\mathcal{M}_{\text{tot}} = \mathcal{M}_t + \mathcal{M}_s \tag{16}$$

The squared amplitude

$$|\mathcal{M}_{\text{tot}}|^2 = |\mathcal{M}_t|^2 + |\mathcal{M}_s|^2 + \mathcal{M}_t\mathcal{M}_s^* + \mathcal{M}_s\mathcal{M}_t^* \tag{17}$$

$$\begin{aligned} = \frac{e^4}{(2\pi)^6 2p_2^0 p_3^0}\Bigg[&\frac{1}{t^2}[\bar{u}_4\not{\epsilon}_2(\not{q}' + m_e)\not{\epsilon}_3^* u_1][\bar{u}_4\not{\epsilon}_2(\not{q}' + m_e)\not{\epsilon}_3^* u_1]^* \\ &+ \frac{1}{s^2}[\bar{u}_4\not{\epsilon}_3^*(\not{q}'' + m_e)\not{\epsilon}_2 u_1][\bar{u}_4\not{\epsilon}_3^*(\not{q}'' + m_e)\not{\epsilon}_2 u_1]^* \\ &+ \frac{1}{ts}[\bar{u}_4\not{\epsilon}_2(\not{q}' + m_e)\not{\epsilon}_3^* u_1][\bar{u}_4\not{\epsilon}_3^*(\not{q}'' + m_e)\not{\epsilon}_2 u_1]^* \\ &+ \frac{1}{ts}[\bar{u}_4\not{\epsilon}_3^*(\not{q}'' + m_e)\not{\epsilon}_2 u_1][\bar{u}_4\not{\epsilon}_2(\not{q}' + m_e)\not{\epsilon}_3^* u_1]^*\Bigg] \end{aligned} \tag{18}$$

We have a bunch of factors of the form

$$[\bar{u}_a\not{\epsilon}_b(\not{q} + m_e)\not{\epsilon}_c^* u_d]^* = [u_a^\dagger\gamma^0(\epsilon_b)_\nu\gamma^\nu q_\mu\gamma^\mu(\epsilon_c)_\rho^*\gamma^\rho u_d]^\dagger + m_e[u_a^\dagger\gamma^0(\epsilon_b)_\nu\gamma^\nu(\epsilon_c)_\rho^*\gamma^\rho u_d]^\dagger \tag{19}$$

$$= [u_d^\dagger(\gamma^\rho)^\dagger(\epsilon_c)_\rho(\gamma^\mu)^\dagger q_\mu(\gamma^\nu)^\dagger(\epsilon_b)_\nu^*(\gamma^0)^\dagger u_a] + m_e[u_d^\dagger(\gamma^\rho)^\dagger(\epsilon_c)_\rho(\gamma^\nu)^\dagger(\epsilon_b)_\nu^*(\gamma^0)^\dagger u_a] \tag{20}$$

$$= [u_d^\dagger\gamma^0\gamma^\rho\gamma^0(\epsilon_c)_\rho\gamma^0\gamma^\mu\gamma^0 q_\mu\gamma^0\gamma^\nu\gamma^0(\epsilon_b)_\nu^*\gamma^0 u_a] + m_e[u_d^\dagger\gamma^0\gamma^\rho\gamma^0(\epsilon_c)_\rho\gamma^0\gamma^\nu\gamma^0(\epsilon_b)_\nu^*\gamma^0 u_a] \tag{21}$$

$$= [\bar{u}_d\not{\epsilon}_c\not{q}\not{\epsilon}_b^* u_a] + m_e[\bar{u}_d\not{\epsilon}_c\not{\epsilon}_b^* u_a] = [\bar{u}_d\not{\epsilon}_c(\not{q} + m_e)\not{\epsilon}_b^* u_a] \tag{22}$$

Where I have used the following facts: $(\gamma^0)^\dagger = \gamma^0, (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$, and the fact that q_μ and ϵ_μ are just numbers (real and complex, respectively). Defining $Q' \equiv \not{q}' + m_e$, $Q'' \equiv \not{q}'' + m_e$ and focusing on the terms within the brakets of Eq.(18)

$$\begin{aligned}&\frac{1}{t^2}[\bar{u}_4\not{\epsilon}_2 Q'\not{\epsilon}_3^* u_1][\bar{u}_1\not{\epsilon}_3 Q'\not{\epsilon}_2^* u_4] + \frac{1}{s^2}[\bar{u}_4\not{\epsilon}_3 Q''\not{\epsilon}_2^* u_1][\bar{u}_1\not{\epsilon}_2 Q''\not{\epsilon}_3^* u_4]\\ &+\frac{1}{ts}\left[[\bar{u}_4\not{\epsilon}_2 Q'\not{\epsilon}_3^* u_1][\bar{u}_1\not{\epsilon}_2^* Q''\not{\epsilon}_3 u_4] + [\bar{u}_4\not{\epsilon}_3 Q''\not{\epsilon}_2^* u_1][\bar{u}_1\not{\epsilon}_3^* Q'\not{\epsilon}_2 u_4]\right]\end{aligned} \tag{23}$$

$$\begin{aligned}=&\frac{1}{(p_1\cdot p_3)^2}[\bar{u}_4\epsilon_{2\mu}(\gamma^\mu 2p_1^\nu - \gamma^\mu\not{p}_3\gamma^\nu)\epsilon_{3\nu}^* u_1][\bar{u}_1\epsilon_{3\eta}(2p_1^\eta\gamma^\lambda - \gamma^\eta\not{p}_3\gamma^\lambda)\epsilon_{2\lambda}^* u_4]\\ &\frac{1}{(p_1\cdot p_2)^2}[\bar{u}_4\epsilon_{3\mu}^*(2\gamma^\mu p_1^\nu + \gamma^\mu\not{p}_2\gamma^\nu)\epsilon_{2\nu} u_1][\bar{u}_1\epsilon_{2\eta}^*(2p_1^\eta\gamma^\lambda + \gamma^\eta\not{p}_2\gamma^\lambda)\epsilon_{3\lambda} u_4]\\ &+\frac{1}{(p_1\cdot p_3)(p_1\cdot p_2)}\Big[[\bar{u}_4\epsilon_{2\mu}(\gamma^\mu 2p_1^\nu - \gamma^\mu\not{p}_3\gamma^\nu)\epsilon_{3\nu}^* u_1][\bar{u}_1\epsilon_{2\nu}^*(2\gamma^\mu p_1^\nu + \gamma^\nu\not{p}_2\gamma^\mu)\epsilon_{3\mu} u_4]\\ &+[\bar{u}_4\epsilon_{3\mu}^*(2\gamma^\mu p_1^\nu + \gamma^\mu\not{p}_2\gamma^\nu)\epsilon_{2\nu} u_1][\bar{u}_1\epsilon_{3\nu}(2p_1^\nu\gamma^\mu - \gamma^\nu\not{p}_3\gamma^\mu)\epsilon_{2\mu}^* u_4]\Big]\end{aligned} \tag{24}$$

Now comes the job of averaging and summing over spins, we have terms of the two forms:

$$\frac{1}{2}\sum_{\sigma_1=\pm1/2}\frac{1}{2}\sum_{\sigma_a=\pm1}\sum_{\sigma_b=\pm1}\sum_{\sigma_4=\pm1/2}[\bar{u}_4\not{\epsilon}_a Q\not{\epsilon}_b^* u_1][\bar{u}_1\not{\epsilon}_b\tilde{Q}\not{\epsilon}_a^* u_4] \tag{25}$$

$$=\frac{1}{4}\sum_{\sigma_4=\pm1/2}\sum_{\sigma_a,\sigma_=\pm1}[\bar{u}_4\not{\epsilon}_a Q\not{\epsilon}_b^*(\not{p}_1 + m_e)\not{\epsilon}_b\tilde{Q}\not{\epsilon}_a^* u_4] \tag{26}$$

$$=\frac{1}{4}\sum_{\sigma_4=\pm1/2}\sum_{\sigma_a,\sigma_b=\pm1}[\bar{u}_4\gamma^\mu Q\gamma^\nu(\not{p}_1 + m_e)\gamma^\rho\tilde{Q}\gamma^\eta u_4](\epsilon_b)_\rho(\epsilon_b)_\nu^*(\epsilon_a)_\mu(\epsilon_a)_\eta^* \tag{27}$$

$$=\frac{1}{4}\sum_{\sigma_4=\pm1/2}[\bar{u}_4\gamma^\mu Q\gamma^\nu(\not{p}_1 + m_e)\gamma^\rho\tilde{Q}\gamma^\eta u_4]g_{\rho\nu}g_{\mu\eta} \tag{28}$$

$$=\frac{1}{4}\sum_{ij}[\gamma^\mu Q\gamma^\nu(\not{p}_1 + m_e)\gamma_\nu\tilde{Q}\gamma_\mu]_{ij}\sum_{\sigma_4=\pm1/2}[\bar{u}_4 u_4]_{ji} \tag{29}$$

$$=\frac{1}{4}\sum_{j}[\gamma^\mu Q\gamma^\nu(\not{p}_1 + m_e)\gamma_\nu\tilde{Q}\gamma_\mu(\not{p}_4 + m_e)]_{jj} \tag{30}$$

$$=\frac{1}{4}\mathrm{Tr}[\gamma^\mu(\not{q} + m_e)\gamma^\nu(\not{p}_1 + m_e)\gamma_\nu(\tilde{\not{q}} + m_e)\gamma_\mu(\not{p}_4 + m_e)] \tag{31}$$

Let's see what comes out when we look at the spin sum and average of Eq.(24). For the first and second term we have the form

$$\frac{1}{4}\sum_{\sigma_4=\pm 1/2}\sum_{\sigma_2,\sigma_3=\pm 1}[\bar{u}_4(\gamma^\mu 2p_1^\nu \mp \gamma^\mu \not{p}_{3/2}\gamma^\nu)(\not{p}_1+m_e)(2p_1^\eta\gamma^\lambda \mp \gamma^\eta \not{p}_{3/2}\gamma^\lambda)u_4](\epsilon_2)_\mu(\epsilon_2)^*_\lambda(\epsilon_3)_\eta(\epsilon_3)^*_\nu \tag{32}$$

$$= \frac{1}{4}\sum_{\sigma_4=\pm 1/2}[\bar{u}_4(\gamma^\mu 2p_1^\nu \mp \gamma^\mu \not{p}_{3/2}\gamma^\nu)(\not{p}_1+m_e)(2p_1^\eta\gamma^\lambda \mp \gamma^\eta \not{p}_{3/2}\gamma^\lambda)u_4]g^{\mu\lambda}g^{\eta\nu} \tag{33}$$

$$= \frac{1}{4}\mathrm{Tr}[(2\gamma^\mu p_1^\nu \mp \gamma^\mu \not{p}_{3/2}\gamma^\nu)(\not{p}_1+m_e)(2p_{1\nu}\gamma_\mu \mp \gamma_\nu \not{p}_{3/2}\gamma_\mu)(\not{p}_4+m_e)] \tag{34}$$

Expanding the traces explicitly we have

$$\begin{aligned}
=& m_e\mathrm{Tr}[\gamma^\mu\gamma^\eta q_\eta\gamma^\nu\gamma^\lambda p_{1\lambda}\gamma_\nu\gamma^\sigma\tilde{q}_\sigma\gamma_\mu] + m_e\mathrm{Tr}[\gamma^\mu\gamma^\eta q_\eta\gamma^\nu\gamma_\nu\gamma^\sigma\tilde{q}_\sigma\gamma_\mu\gamma^\rho p_{4\rho}]\\
&+ m_e\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda p_{1\lambda}\gamma_\nu\gamma^\sigma\tilde{q}_\sigma\gamma_\mu\gamma^\rho p_{4\rho}] + m_e^2\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda p_{1\lambda}\gamma_\nu\gamma^\sigma\tilde{q}_\sigma\gamma_\mu]\\
&+ m_e^2\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma_\nu\gamma^\sigma\tilde{q}_\sigma\gamma_\mu\gamma^\rho p_{4\rho}] + m_e^2\mathrm{Tr}[\gamma^\mu\gamma^\eta q_\eta\gamma^\nu\gamma_\nu\gamma^\sigma\tilde{q}_\sigma\gamma_\mu]\\
&+ m_e^3\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma_\nu\gamma^\sigma\tilde{q}_\sigma\gamma_\mu] + \mathrm{Tr}[\gamma^\mu\gamma^\eta q_\eta\gamma^\nu\gamma^\lambda p_{1\lambda}\gamma_\nu\gamma^\sigma\tilde{q}_\sigma\gamma_\mu\gamma^\rho p_{4\rho}]\\
&+ m_e\mathrm{Tr}[\gamma^\mu\gamma^\eta q_\eta\gamma^\nu\gamma^\lambda p_{1\lambda}\gamma_\nu\gamma_\mu\gamma^\rho p_{4\rho}] + m_e^2\mathrm{Tr}[\gamma^\mu\gamma^\eta q_\eta\gamma^\nu\gamma^\lambda p_{1\lambda}\gamma_\nu\gamma_\mu]\\
&+ m_e^2\mathrm{Tr}[\gamma^\mu\gamma^\eta q_\eta\gamma^\nu\gamma_\nu\gamma_\mu\gamma^\rho p_{4\rho}] + m_e^2\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda p_{1\lambda}\gamma_\nu\gamma_\mu\gamma^\rho p_{4\rho}]\\
&+ m_e^3\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda p_{1\lambda}\gamma_\nu\gamma_\mu] + m_e^3\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma_\nu\gamma_\mu\gamma^\rho p_{4\rho}]\\
&+ m_e^3\mathrm{Tr}[\gamma^\mu\gamma^\eta q_\eta\gamma^\nu\gamma_\nu\gamma_\mu] + m_e^4\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma_\nu\gamma_\mu]
\end{aligned} \tag{35}$$

$$\begin{aligned}
=& m_e q_\eta p_{1\lambda}\tilde{q}_\sigma\mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\nu\gamma^\lambda\gamma_\nu\gamma^\sigma\gamma_\mu] + m_e q_\eta\tilde{q}_\sigma p_{4\rho}\mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\nu\gamma_\nu\gamma^\sigma\gamma_\mu\gamma^\rho]\\
&+ m_e p_{1\lambda}\tilde{q}_\sigma p_{4\rho}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\nu\gamma^\sigma\gamma_\mu\gamma^\rho] + m_e^2 p_{1\lambda}\tilde{q}_\sigma\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\nu\gamma^\sigma\gamma_\mu]\\
&+ m_e^2\tilde{q}_\sigma p_{4\rho}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma_\nu\gamma^\sigma\gamma_\mu\gamma^\rho] + m_e^2 q_\eta\tilde{q}_\sigma\mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\nu\gamma_\nu\gamma^\sigma\gamma_\mu]\\
&+ m_e^3\tilde{q}_\sigma\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma_\nu\gamma^\sigma\gamma_\mu] + q_\eta p_{1\lambda}\tilde{q}_\sigma p_{4\rho}\mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\nu\gamma^\lambda\gamma_\nu\gamma^\sigma\gamma_\mu\gamma^\rho]\\
&+ m_e q_\eta p_{1\lambda} p_{4\rho}\mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\nu\gamma^\lambda\gamma_\nu\gamma_\mu\gamma^\rho] + m_e^2 q_\eta p_{1\lambda}\mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\nu\gamma^\lambda\gamma_\nu\gamma_\mu]\\
&+ m_e^2 q_\eta p_{4\rho}\mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\nu\gamma_\nu\gamma_\mu\gamma^\rho] + m_e^2 p_{1\lambda}p_{4\rho}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\nu\gamma_\mu\gamma^\rho]\\
&+ m_e^3 p_{1\lambda}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\nu\gamma_\mu] + m_e^3 p_{4\rho}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma_\nu\gamma_\mu\gamma^\rho]\\
&+ m_e^3 q_\eta\mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\nu\gamma_\nu\gamma_\mu] + m_e^4\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma_\nu\gamma_\mu]
\end{aligned} \tag{36}$$

We can simplify first by using the fact that the trace of a product of an odd number of γ matrices is zero

$$
\begin{aligned}
=& m_e^2 p_{1\lambda}\tilde{q}_\sigma \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\nu\gamma^\sigma\gamma_\mu] + m_e^2\tilde{q}_\sigma p_{4\rho}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma_\nu\gamma^\sigma\gamma_\mu\gamma^\rho] + m_e^2 q_\eta\tilde{q}_\sigma\mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\nu\gamma_\nu\gamma^\sigma\gamma_\mu] \\
&+ q_\eta p_{1\lambda}\tilde{q}_\sigma p_{4\rho}\mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\nu\gamma^\lambda\gamma_\nu\gamma^\sigma\gamma_\mu\gamma^\rho] + m_e^2 q_\eta p_{1\lambda}\mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\nu\gamma^\lambda\gamma_\nu\gamma_\mu] \\
&+ m_e^2 q_\eta p_{4\rho}\mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\nu\gamma_\nu\gamma_\mu\gamma^\rho] + m_e^2 p_{1\lambda}p_{4\rho}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\nu\gamma_\mu\gamma^\rho] + m_e^4\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma_\nu\gamma_\mu]
\end{aligned} \tag{37}
$$

Or for the simplified expression for $q = p_1 + p_2$, leaving out the odd numbered gamma matrix products

$$
\begin{aligned}
=& 4m_e^2(p_1\cdot p_1)\mathrm{Tr}[\gamma^\mu\gamma_\mu] + 2m_e^2 p_{2\sigma}p_{1\nu}\mathrm{Tr}[\gamma^\mu\gamma^\sigma\gamma^\nu\gamma_\mu] \\
&+ 4(p_1\cdot p_1)p_{1\lambda}p_{4\eta}\mathrm{Tr}[\gamma^\mu\gamma^\lambda\gamma_\mu\gamma^\eta] + 2p_{2\sigma}p_{1\lambda}p_{1\nu}p_{4\eta}\mathrm{Tr}[\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\lambda\gamma_\mu\gamma^\eta] \\
&+ 2m_e^2 p_1^\nu p_{2\rho}\mathrm{Tr}[\gamma^\mu\gamma_\nu\gamma^\rho\gamma_\mu] + m_e^2 p_{2\sigma}p_{2\rho}\mathrm{Tr}[\gamma^\mu\gamma^\sigma\gamma^\nu\gamma_\nu\gamma^\rho\gamma_\mu] \\
&+ 2p_1^\nu p_{1\lambda}p_{2\rho}p_{4\eta}\mathrm{Tr}[\gamma^\mu\gamma^\lambda\gamma_\nu\gamma^\rho\gamma_\mu\gamma^\eta] + p_{2\sigma}p_{1\lambda}p_{2\rho}p_{4\eta}\mathrm{Tr}[\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\lambda\gamma_\nu\gamma^\rho\gamma_\mu\gamma^\eta]
\end{aligned} \tag{38}
$$

Before we continue on we should note that we will probably want to express our matrix elements in terms of Lorentz invariant quantities, for $2 \to 2$ scattering we can express our result in the Lorentz-invariant Mandelstama variables. We can choose any frame we'd like to define them, in the CM frame we have

$$
\begin{aligned}
s &= (p_1+p_2)^2 = 2(p_1\cdot p_2) + m_e^2 \\
&= (p_3+p_4)^2 = 2(p_3\cdot p_4) + m_e^2
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
t &= (p_1-p_3)^2 = -2(p_1\cdot p_3) + m_e^2 \\
&= (p_4-p_2)^2 = -2(p_4\cdot p_2) + m_e^2
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
u &= (p_1-p_4)^2 = -2(p_1\cdot p_4) + 2m_e^2 \\
&= (p_3-p_2)^2 = -2(p_3\cdot p_2)
\end{aligned} \tag{41}
$$

Now we can go term by term, simplifying with our γ-matrix identities

1.

$$
m_e^2 p_{1\lambda}\tilde{q}_\sigma\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\nu\gamma^\sigma\gamma_\mu] = -2m_e^2 p_{1\lambda}\tilde{q}_\sigma\mathrm{Tr}[\gamma^\mu\gamma^\lambda\gamma^\sigma\gamma_\mu] = -32m_e^2 p_{1\lambda}\tilde{q}_\sigma g^{\lambda\sigma} \tag{42}
$$

$$
= -32m_e^2(p_1\cdot\tilde{q}) \tag{43}
$$

Where I have used the identities $\gamma^\nu\gamma^\lambda\gamma_\nu = -2\gamma^\lambda$, $\gamma^\mu\gamma^\lambda\gamma^\sigma\gamma_\mu = 4g^{\lambda\sigma}\mathbb{1}_{4\times4}$.

For $\tilde{q} = p_1 - p_3$

$$
= -32m_e^2(p_1^2 - (p_1\cdot p_3)) = -32m_e^2\left(m_e^2 - \left(\frac{m_e^2}{2} - \frac{t}{2}\right)\right) = -16m_e^4 - 16m_e^2 t \tag{44}
$$

For $\tilde{q} = p_1 + p_2$

$$= -32m_e^2(p_1^2 + p_1 \cdot p_2) = -32m_e^2\left(m_e^2 + \left(\frac{s}{2} - \frac{m_e^2}{2}\right)\right) = -16m_e^4 - 16m_e^2 s \quad (45)$$

2.

$$m_e^2 \tilde{q}_\sigma p_{4_\rho} \mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma_\nu \gamma^\sigma \gamma_\mu \gamma^\rho] = 4m_e^2 \tilde{q}_\sigma p_{4_\rho} \mathrm{Tr}[\gamma^\mu \gamma^\sigma \gamma_\mu \gamma^\rho] = -8m_e^2 \tilde{q}_\sigma p_{4_\rho} \mathrm{Tr}[\gamma^\sigma \gamma^\rho] \quad (46)$$

$$= -32m_e^2 \tilde{q}_\sigma p_{4_\rho} g^{\sigma\rho} = -32m_e^2\,(\tilde{q} \cdot p_4) \quad (47)$$

Using $\gamma^\nu \gamma_\nu = 4\mathbb{1}_{4\times 4}$, $\mathrm{Tr}[\gamma^\sigma \gamma^\rho] = 4g^{\sigma\rho}$

For $\tilde{q} = p_1 - p_3$

$$= -32m_e^2((p_1 \cdot p_4) - (p_3 \cdot p_4)) = -32m_e^2\left(-\frac{u}{2} + m_e^2 - \frac{s}{2} + \frac{m_e^2}{2}\right) = 16m_e^2 u - 16m_e^4 - 16m_e^2 s \quad (48)$$

For $\tilde{q} = p_1 + p_2$

$$= -32m_e^2((p_1 \cdot p_4) + (p_2 \cdot p_4)) = -32m_e^2\left(-\frac{u}{2} + m_e^2 + \frac{t}{2} - \frac{m_e^2}{2}\right) = -48m_e^4 + 16m_e^2 t + 16m_e^2 u \quad (49)$$

3.

$$m_e^2 q_\eta \tilde{q}_\sigma \mathrm{Tr}[\gamma^\mu \gamma^\eta \gamma^\nu \gamma_\nu \gamma^\sigma \gamma_\mu] = 64m_e^2 q_\eta \tilde{q}_\sigma g^{\eta\sigma} = 64m_e^2(q \cdot \tilde{q}) \quad (50)$$

For $q = \tilde{q} = p_1 - p_3$

$$= 64m_e^2(p_1 - p_3)^2 = 64m_e^2 t \quad (51)$$

For $q = \tilde{q} = p_1 + p_2$

$$= 64m_e^2((p_1 + p_2)^2) = 64m_e^2 s \quad (52)$$

4.

$$q_\eta p_{1\lambda} \tilde{q}_\sigma p_{4_\rho} \mathrm{Tr}[\gamma^\mu \gamma^\eta \gamma^\nu \gamma^\lambda \gamma_\nu \gamma^\sigma \gamma_\mu \gamma^\rho] = -2q_\eta p_{1\lambda} \tilde{q}_\sigma p_{4_\rho} \mathrm{Tr}[\gamma^\mu \gamma^\eta \gamma^\lambda \gamma^\sigma \gamma_\mu \gamma^\rho] \quad (53)$$

$$= 4q_\eta p_{1\lambda} \tilde{q}_\sigma p_{4_\rho} \mathrm{Tr}[\gamma^\sigma \gamma^\lambda \gamma^\eta \gamma^\rho] = 16q_\eta p_{1\lambda} \tilde{q}_\sigma p_{4_\rho}(g^{\sigma\lambda} g^{\eta\rho} - g^{\sigma\eta} g^{\lambda\rho} + g^{\sigma\rho} g^{\lambda\eta}) \quad (54)$$

$$= 16\left[(p_1 \cdot \tilde{q})(q \cdot p_4) - (q \cdot \tilde{q})(p_1 \cdot p_4) + (\tilde{q} \cdot p_4)(p_1 \cdot q)\right] \quad (55)$$

For $q = \tilde{q}$ this reduces to

$$= 32(p_1 \cdot q)(p_4 \cdot q) - 16q^2(p_1 \cdot p_4) \quad (56)$$

For $q = p_1 - p_3$

$$= 8(p_1 \cdot (p_1 - p_3)p_4 \cdot (p_1 - p_3)) - 4(p_1 - p_3)^2(p_1 \cdot p_4) \tag{57}$$

$$= 8([p_1^2 - (p_1 \cdot p_3)][(p_1 \cdot p_4) - (p_3 \cdot p_4)]) - 4t(p_1 \cdot p_4) \tag{58}$$

$$= 8\left(\left[m_e^2 - \frac{m_e^2}{2} + \frac{t}{2}\right]\left[-\frac{u}{2} + m_e^2 - \frac{s}{2} + \frac{m_e^2}{2}\right]\right) - 4t\left(-\frac{u}{2} + m_e^2\right) \tag{59}$$

$$= 2\left(m_e^2(s - t - u) + m_e^4 + st\right) \tag{60}$$

For $q = p_1 + p_2$

$$= 8(p_1 \cdot (p_1 + p_2)p_4 \cdot (p_1 + p_2)) - 4(p_1 + p_2)^2(p_1 \cdot p_4) \tag{61}$$

$$= 8([p_1^2 + (p_1 \cdot p_2)][(p_1 \cdot p_4) + (p_2 \cdot p_4)]) - 4s(p_1 \cdot p_4) \tag{62}$$

$$= 8\left(\left[m_e^2 + \frac{s}{2} - \frac{m_e^2}{2}\right]\left[-\frac{u}{2} + m_e^2 - \frac{t}{2} + \frac{m_e^2}{2}\right]\right) - 4s\left(-\frac{u}{2} + m_e^2\right) \tag{63}$$

$$= 6m_e^4 - 2st + 2m_e^2(s - t - u) \tag{64}$$

5.

$$m_e^2 q_\eta p_{1\lambda} \mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\nu\gamma^\lambda\gamma_\nu\gamma_\mu] = -2m_e^2 q_\eta p_{1\lambda} \mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\lambda\gamma_\mu] \tag{65}$$

$$= -32m_e^2 q_\eta p_{1\lambda} g^{\eta\lambda} = -32m_e^2(q \cdot p_1) \tag{66}$$

6.

$$m_e^2 q_\eta p_{4\rho} \mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\nu\gamma_\nu\gamma_\mu\gamma^\rho] = 4m_e^2 q_\eta p_{4\rho} \mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma_\mu\gamma^\rho] = -8m_e^2 q_\eta p_{4\rho} \mathrm{Tr}[\gamma^\eta\gamma^\rho] \tag{67}$$

$$= -32m_e^2 q_\eta p_{4\rho} g^{\eta\rho} = -32m_e^2(q \cdot p_4) \tag{68}$$

7.

$$m_e^2 p_{1\lambda} p_{4\rho} \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\nu\gamma_\mu\gamma^\rho] = -2m_e^2 p_{1\lambda} p_{4\rho} \mathrm{Tr}[\gamma^\mu\gamma^\lambda\gamma_\mu\gamma^\rho] = 4m_e^2 p_{1\lambda} p_{4\rho} \mathrm{Tr}[\gamma^\lambda\gamma^\rho] \tag{69}$$

$$= 16m_e^2 p_{1\lambda} p_{4\rho} g^{\lambda\rho} = 16m_e^2(p_1 \cdot p_4) \tag{70}$$

In terms of Mandelstam variables

$$= 16m_e^2\left[-\frac{u}{2} + m_e^2\right] = -8m_e^2 u + 16m_e^4 \tag{71}$$

8.

$$m_e^4 \mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma_\nu \gamma_\mu] = 64 m_e^4 \tag{72}$$

Putting it all together,

$$\begin{aligned}
&\mathrm{Tr}[\gamma^\mu(\not{q} + m_e)\gamma^\nu(\not{p}_1 + m_e)\gamma_\nu(\not{\tilde{q}} + m_e)\gamma_\mu(\not{p}_4 + m_e)] \\
&\quad = -32 m_e^2 (p_1 \cdot \tilde{q}) - 32 m_e^2 (\tilde{q} \cdot p_4) + 64 m_e^2 (q \cdot \tilde{q}) \\
&\quad + 4\left[(p_1 \cdot \tilde{q})(q \cdot p_4) - (q \cdot \tilde{q})(p_1 \cdot p_4) + (\tilde{q} \cdot p_4)(p_1 \cdot q)\right] \\
&\quad - 32 m_e^2 (q \cdot p_1) - 32 m_e^2 (q \cdot p_4) + 16 m_e^2 (p_1 \cdot p_4) + 64 m_e^4
\end{aligned} \tag{73}$$

$$\begin{aligned}
=&64 m_e^4 - 32 m_e^2 [(p_1 \cdot \tilde{q}) + (\tilde{q} \cdot p_4) + (q \cdot p_1) + (q \cdot p_4)] + 64 m_e^2 (q \cdot \tilde{q}) \\
&+ 16 m_e^2 (p_1 \cdot p_4) + 4\left[(p_1 \cdot \tilde{q})(q \cdot p_4) - (q \cdot \tilde{q})(p_1 \cdot p_4) + (\tilde{q} \cdot p_4)(p_1 \cdot q)\right]
\end{aligned} \tag{74}$$

For $q = \tilde{q} = p_1 - p_3$

$$\begin{aligned}
=& - 16 m_e^4 - 16 m_e^2 t + 16 m_e^2 u - 48 m_e^4 + 16 m_e^2 s + 64 m_e^2 t + 6 m_e^4 - 2 m_e^2 (u - t + s) - 2ts \\
&- 16 m_e^4 - 16 m_e^2 t + 16 m_e^2 u - 48 m_e^4 + 16 m_e^2 s - 8 m_e^2 u + 16 m_e^4 + 64 m_e^4
\end{aligned} \tag{75}$$

$$= -24 m^2 s + 24 m^2 t + 16 m^2 u + 24 m^4 + 8st \tag{76}$$

$$= 8(m^2(-3s + 3t + 2u) + 3m^4 + st) \tag{77}$$

The simplified expression i.e. Eq.(38) is determined in the same way

1.

$$4 m_e^2 (p_1 \cdot p_1) \mathrm{Tr}[\gamma^\mu \gamma_\mu] = 64 m_e^4 \tag{78}$$

2.

$$2 m_e^2 p_{2\sigma} p_{1\nu} \mathrm{Tr}[\gamma^\mu \gamma^\sigma \gamma^\nu \gamma_\mu] = 32 m_e^2 (p_1 \cdot p_2) \tag{79}$$

$$= 16 m_e^2 (s - m_e^2) \tag{80}$$

3.

$$4(p_1 \cdot p_1) p_{1\lambda} p_{4\eta} \mathrm{Tr}[\gamma^\mu \gamma^\lambda \gamma_\mu \gamma^\eta] = -32 (p_1 \cdot p_1)(p_1 \cdot p_4) \tag{81}$$

$$= 16 m_e^2 u - 32 m_e^4 \tag{82}$$

4.

$$2 p_{2\sigma} p_{1\lambda} p_{1\nu} p_{4\eta} \mathrm{Tr}[\gamma^\mu \gamma^\sigma \gamma^\nu \gamma^\lambda \gamma_\mu \gamma^\eta] = -16[(p_2 \cdot p_4)(p_1 \cdot p_1) - (p_1 \cdot p_4)(p_1 \cdot p_2) + (p_1 \cdot p_4)(p_1 \cdot p_2))] \tag{83}$$

$$= -16 m_e^2 (p_2 \cdot p_4) = 8 m_e^2 t - 8 m_e^4 \tag{84}$$

5.

$$2m_e^2 p_1^\nu p_{2\rho} \mathrm{Tr}[\gamma^\mu \gamma_\nu \gamma^\rho \gamma_\mu] = 32m_e^2(p_1 \cdot p_2) \tag{85}$$
$$= 16(m_e^2 s - m_e^4) \tag{86}$$

6.

$$m_e^2 p_{2\sigma} p_{2\rho} \mathrm{Tr}[\gamma^\mu \gamma^\sigma \gamma^\nu \gamma_\nu \gamma^\rho \gamma_\mu] = 64m_e^2(p_2 \cdot p_2) = 0 \tag{87}$$

7.

$$2p_1^\nu p_{1\lambda} p_{2\rho} p_{4\eta} \mathrm{Tr}[\gamma^\mu \gamma^\lambda \gamma_\nu \gamma^\rho \gamma_\mu \gamma^\eta] = -16[(p_2 \cdot p_4)(p_1 \cdot p_1) - (p_1 \cdot p_4)(p_1 \cdot p_2) + (p_1 \cdot p_4)(p_1 \cdot p_2)] \tag{88}$$
$$= -16m_e^2(p_2 \cdot p_4) = 8m_e^2 t - 8m_e^4 \tag{89}$$

8.

$$p_{2\sigma} p_{1\lambda} p_{2\rho} p_{4\eta} \mathrm{Tr}[\gamma^\mu \gamma^\sigma \gamma^\nu \gamma^\lambda \gamma_\nu \gamma^\rho \gamma_\mu \gamma^\eta] = 16[(p_2 \cdot p_4)(p_1 \cdot p_2) + (p_2 \cdot p_4)(p_1 \cdot p_2) - (p_1 \cdot p_4)(p_2 \cdot p_2)] \tag{90}$$
$$= 32(p_2 \cdot p_4)(p_1 \cdot p_2) = 8m_e^2 s + 8m_e^2 t - 8m_e^4 - 8st \tag{91}$$

Now we can evaluate the first two terms in Eq.(23). For the first term we realize that we should have set $Q = \tilde{Q}$ above which would have saved us some time, but, nonetheless we push on setting $q = \tilde{q}$

$$= 64m_e^4 - 64m_e^2[(p_1 \cdot q) + (p_4 \cdot q) - q^2] + 16m_e^2(p_1 \cdot p_4) + 8(p_1 \cdot q)(p_4 \cdot q) - 4q^2(p_1 \cdot p_4) \tag{92}$$

In the first term we note that $q' = p_1 - p_3$

$$\begin{aligned} &\frac{1}{t^2}[\bar{u}_4 \not{\epsilon}_2 Q' \not{\epsilon}_3^* u_1][\bar{u}_1 \not{\epsilon}_3 Q' \not{\epsilon}_2^* u_4] \\ &= \frac{1}{4t^2}\Bigg[64m_e^4 - 64m_e^2[(p_1 \cdot (p_1 - p_3)) + (p_4 \cdot (p_1 - p_3)) - (p_1 - p_3)^2] + 16m_e^2(p_1 \cdot p_4) \\ &+ 8(p_1 \cdot (p_1 - p_3))(p_4 \cdot (p_1 - p_3)) - 4(p_1 - p_3)^2(p_1 \cdot p_4)\Bigg] \end{aligned} \tag{93}$$

$$= \frac{1}{4t^2}\left[64m_e^4 + 16m_e^2(p_1 \cdot p_4) - 64m_e^2\,(p_1 - p_3)\,(p_3 + p_4) + 4(p_1 \cdot p_4)(p_1 - p_3)^2\right] \tag{94}$$

$$= \frac{1}{t^2}\left[-16m^2\sqrt{st} + m^2 t - 2m^2 u + 20m^4 - \frac{tu}{2}\right] \tag{95}$$

For the second terms in Eq.(23) noting $q'' = p_1 + p_2$

$$\frac{1}{s^2}[\bar{u}_4 \not\epsilon_3 Q'' \not\epsilon_2^* u_1][\bar{u}_1 \not\epsilon_2 Q'' \not\epsilon_3^* u_4]$$
$$= \frac{1}{4s^2}\Bigg[64m_e^4 - 64m_e^2[(p_1 \cdot (p_1+p_2)) + (p_4 \cdot (p_1+p_2)) - (p_1+p_2)^2] + 16m_e^2(p_1 \cdot p_4)$$
$$+ 8(p_1 \cdot (p_1+p_2))(p_4 \cdot (p_1+p_2)) - 4(p_1+p_2)^2(p_1 \cdot p_4)\Bigg] \tag{96}$$

$$= \frac{1}{4s^2}\left[64m_e^4 + 16m_e^2(p_1 \cdot p_4) + 64m_e^2(p_1+p_2)(p_2-p_4) + 4(p_1 \cdot p_4)(p_1+p_2)^2\right] \tag{97}$$

$$= 24m^2 s + 8m^2 t + 16m^2 u + 8m^4 + 8st \tag{98}$$

$$= 8\left(m^2(3s+t+2u) + m^4 + st\right) \tag{99}$$

$$= 8\left(2m^2(2s+t+u) + \left(s-m^2\right)\left(t-m^2\right)\right) \tag{100}$$

Thus,

$$\langle|\mathcal{M}_s|^2\rangle = \frac{e^4}{(2\pi)^6 2p_2^0 p_3^0}\frac{1}{8(p_1 \cdot p_2)}[8\left(2m^2(2s+t+u) + \left(s-m^2\right)\left(t-m^2\right)\right)] \tag{101}$$

$$= \frac{e^4}{(2\pi)^6 p_2^0 p_3^0 (p_1 \cdot p_2)}(2m_e^4 + m_e^2(s-m_e^2) - \frac{1}{2}(s-m_e^2)(t-m_e^2)) \tag{102}$$

We can also note that the two amplitudes are related via the transformation $p_2 \to -p_3$ thus,

$$\langle|\mathcal{M}_t|^2\rangle = \frac{-e^4}{(2\pi)^6 p_2^0 p_3^0 (p_1 \cdot p_3)}(2m_e^4 + m_e^2(t-m_e^2) - \frac{1}{2}(s-m_e^2)(t-m_e^2)) \tag{103}$$

Now we need to check and see if the cross terms in Eq.(23) yield different results after we sum and average over spin states. I suspect that it will slightly change the traces due to the different contractions of the polarization vectors. The terms are of the form

$$\frac{1}{2}\sum_{\sigma_1=\pm 1/2}\frac{1}{2}\sum_{\sigma_a=\pm 1}\sum_{\sigma_b=\pm 1}\sum_{\sigma_4=\pm 1/2}[\bar{u}_4 \not\epsilon_a Q \not\epsilon_b^* u_1][\bar{u}_1 \not\epsilon_a^* \tilde{Q} \not\epsilon_b u_4] \tag{104}$$

$$= \frac{1}{4}\sum_{\sigma_4=\pm 1/2}\sum_{\sigma_a,\sigma_=\pm 1}[\bar{u}_4 \not\epsilon_a Q \not\epsilon_b^* (\not p_1 + m_e) \not\epsilon_a^* \tilde{Q} \not\epsilon_b u_4] \tag{105}$$

$$= \frac{1}{4} \sum_{\sigma_4=\pm 1/2} \sum_{\sigma_a,\sigma_b=\pm 1} [\bar{u}_4 \gamma^\mu Q \gamma^\nu (\not{p}_1 + m_e) \gamma^\rho \tilde{Q} \gamma^\eta u_4] (\epsilon_b)_\eta (\epsilon_b)^*_\nu (\epsilon_a)_\mu (\epsilon_a)^*_\rho \tag{106}$$

$$= \frac{1}{4} \sum_{\sigma_4=\pm 1/2} [\bar{u}_4 \gamma^\mu Q \gamma^\nu (\not{p}_1 + m_e) \gamma^\rho \tilde{Q} \gamma^\eta u_4] g_{\eta\nu} g_{\mu\rho} \tag{107}$$

$$= \frac{1}{4} \sum_{ij} [\gamma^\mu Q \gamma^\nu (\not{p}_1 + m_e) \gamma_\mu \tilde{Q} \gamma_\nu]_{ij} \sum_{\sigma_4=\pm 1/2} [\bar{u}_4 u_4]_{ji} \tag{108}$$

$$= \frac{1}{4} \sum_{j} [\gamma^\mu Q \gamma^\nu (\not{p}_1 + m_e) \gamma_\mu \tilde{Q} \gamma_\nu (\not{p}_4 + m_e)]_{jj} \tag{109}$$

$$= \frac{1}{4} \mathrm{Tr}[\gamma^\mu (\not{q} + m_e) \gamma^\nu (\not{p}_1 + m_e) \gamma_\mu (\tilde{\not{q}} + m_e) \gamma_\nu (\not{p}_4 + m_e)] \tag{110}$$

Expanding the trace explicitly again

$$\begin{aligned}
=& m_e \mathrm{Tr}[\gamma^\mu \gamma^\eta q_\eta \gamma^\nu \gamma^\lambda p_{1\lambda} \gamma_\mu \gamma^\sigma \tilde{q}_\sigma \gamma_\nu m] + m_e \mathrm{Tr}[\gamma^\mu \gamma^\eta q_\eta \gamma^\nu \gamma_\mu \gamma^\sigma \tilde{q}_\sigma \gamma_\nu \gamma^\rho p_{4\rho}] \\
&+ m_e \mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\lambda p_{1\lambda} \gamma_\mu \gamma^\sigma \tilde{q}_\sigma \gamma_\nu \gamma^\rho p_{4\rho}] + m_e^2 \mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\lambda p_{1\lambda} \gamma_\mu \gamma^\sigma \tilde{q}_\sigma \gamma_\nu] \\
&+ m_e^2 \mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma_\mu \gamma^\sigma \tilde{q}_\sigma \gamma_\nu \gamma^\rho p_{4\rho}] + m_e^2 \mathrm{Tr}[\gamma^\mu \gamma^\eta q_\eta \gamma^\nu \gamma_\mu \gamma^\sigma \tilde{q}_\sigma \gamma_\nu] \\
&+ m_e^3 \mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma_\mu \gamma^\sigma \tilde{q}_\sigma \gamma_\nu] + \mathrm{Tr}[\gamma^\mu \gamma^\eta q_\eta \gamma^\nu \gamma^\lambda p_{1\lambda} \gamma_\mu \gamma^\sigma \tilde{q}_\sigma \gamma_\nu \gamma^\rho p_{4\rho}] \\
&+ m_e \mathrm{Tr}[\gamma^\mu \gamma^\eta q_\eta \gamma^\nu \gamma^\lambda p_{1\lambda} \gamma_\mu \gamma_\nu \gamma^\rho p_{4\rho}] + m_e^2 \mathrm{Tr}[\gamma^\mu \gamma^\eta q_\eta \gamma^\nu \gamma^\lambda p_{1\lambda} \gamma_\mu \gamma_\nu] \\
&+ m_e^2 \mathrm{Tr}[\gamma^\mu \gamma^\eta q_\eta \gamma^\nu \gamma_\mu \gamma_\nu \gamma^\rho p_{4\rho}] + m_e \mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\lambda p_{1\lambda} \gamma_\mu \gamma_\nu \gamma^\rho p_{4\rho}] \\
&+ m_e^3 \mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\lambda p_{1\lambda} \gamma_\mu \gamma_\nu] + m_e^3 \mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma_\mu \gamma_\nu \gamma^\rho p_{4\rho}] \\
&+ m_e^3 \mathrm{Tr}[\gamma^\mu \gamma^\eta q_\eta \gamma^\nu \gamma_\mu \gamma_\nu] + m_e^4 \mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma_\mu \gamma_\nu]
\end{aligned} \tag{111}$$

$$\begin{aligned}
=& m_e q_\eta p_{1\lambda} \tilde{q}_\sigma \mathrm{Tr}[\gamma^\mu \gamma^\eta \gamma^\nu \gamma^\lambda \gamma_\mu \gamma^\sigma \gamma_\nu] + m_e q_\eta \tilde{q}_\sigma p_{4\rho} \mathrm{Tr}[\gamma^\mu \gamma^\eta \gamma^\nu \gamma_\mu \gamma^\sigma \gamma_\nu \gamma^\rho] \\
&+ m_e p_{1\lambda} \tilde{q}_\sigma p_{4\rho} \mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\lambda \gamma_\mu \gamma^\sigma \gamma_\nu \gamma^\rho] + m_e^2 p_{1\lambda} \tilde{q}_\sigma \mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\lambda \gamma_\mu \gamma^\sigma \gamma_\nu] \\
&+ m_e^2 \tilde{q}_\sigma p_{4\rho} \mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma_\mu \gamma^\sigma \gamma_\nu \gamma^\rho] + m_e^2 q_\eta \tilde{q}_\sigma \mathrm{Tr}[\gamma^\mu \gamma^\eta \gamma^\nu \gamma_\mu \gamma^\sigma \gamma_\nu] \\
&+ m_e^3 \tilde{q}_\sigma \mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma_\mu \gamma^\sigma \gamma_\nu] + q_\eta p_{1\lambda} \tilde{q}_\sigma p_{4\rho} \mathrm{Tr}[\gamma^\mu \gamma^\eta \gamma^\nu \gamma^\lambda \gamma_\mu \gamma^\sigma \gamma_\nu \gamma^\rho] \\
&+ m_e q_\eta p_{1\lambda} p_{4\rho} \mathrm{Tr}[\gamma^\mu \gamma^\eta \gamma^\nu \gamma^\lambda \gamma_\mu \gamma_\nu \gamma^\rho] + m_e^2 q_\eta p_{1\lambda} \mathrm{Tr}[\gamma^\mu \gamma^\eta \gamma^\nu \gamma^\lambda \gamma_\mu \gamma_\nu] \\
&+ m_e^2 q_\eta p_{4\rho} \mathrm{Tr}[\gamma^\mu \gamma^\eta \gamma^\nu \gamma_\mu \gamma_\nu \gamma^\rho] + m_e^2 p_{1\lambda} p_{4\rho} \mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\lambda \gamma_\mu \gamma_\nu \gamma^\rho] \\
&+ m_e^3 p_{1\lambda} \mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\lambda \gamma_\mu \gamma_\nu] + m_e^3 p_{4\rho} \mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma_\mu \gamma_\nu \gamma^\rho] \\
&+ m_e^3 q_\eta \mathrm{Tr}[\gamma^\mu \gamma^\eta \gamma^\nu \gamma_\mu \gamma_\nu] + m_e^4 \mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma_\mu \gamma_\nu]
\end{aligned} \tag{112}$$

Now we can evaluate the traces. Once again the terms with a product off an odd number of γ matrices is zero, we're left with

$$
\begin{aligned}
=&m_e^2 p_{1\lambda}\tilde{q}_\sigma \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\mu\gamma^\sigma\gamma_\nu] + m_e^2\tilde{q}_\sigma p_{4\rho}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma_\mu\gamma^\sigma\gamma_\nu\gamma^\rho] + m_e^2 q_\eta\tilde{q}_\sigma\mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\nu\gamma_\mu\gamma^\sigma\gamma_\nu] \\
&+ q_\eta p_{1\lambda}\tilde{q}_\sigma p_{4\rho}\mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\nu\gamma^\lambda\gamma_\mu\gamma^\sigma\gamma_\nu\gamma^\rho] + m_e^2 q_\eta p_{1\lambda}\mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\nu\gamma^\lambda\gamma_\mu\gamma_\nu] \\
&+ m_e^2 q_\eta p_{4\rho}\mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\nu\gamma_\mu\gamma_\nu\gamma^\rho] + m_e^2 p_{1\lambda}p_{4\rho}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\mu\gamma_\nu\gamma^\rho] + m_e^4\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma_\mu\gamma_\nu]
\end{aligned}
\tag{113}
$$

Going term by term

1.
$$m_e^2 p_{1\lambda}\tilde{q}_\sigma\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\mu\gamma^\sigma\gamma_\nu] = 4m_e^2 p_{1\lambda}\tilde{q}_\sigma\mathrm{Tr}[\gamma^\sigma\gamma^\lambda] = 16m_e^2 p_{1\lambda}\tilde{q}_\sigma g^{\sigma\lambda} \tag{114}$$
$$= 16m_e^2(p_1\cdot\tilde{q}) \tag{115}$$

For $\tilde{q} = p_1 - p_3$
$$= 8m_e^4 + 8m_e^2 t \tag{116}$$

For $\tilde{p} = p_1 + p_2$
$$= 8m_e^4 + 8m_e^2 s \tag{117}$$

2.
$$m_e^2\tilde{q}_\sigma p_{4\rho}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma_\mu\gamma^\sigma\gamma_\nu\gamma^\rho] = -2m_e^2\tilde{q}_\sigma p_{4\rho}\mathrm{Tr}[\gamma^\nu\gamma^\sigma\gamma_\nu\gamma^\rho] = 4m_e^2\tilde{q}_\sigma p_{4\rho}\mathrm{Tr}[\gamma^\sigma\gamma^\rho] \tag{118}$$
$$= 16m_e^2\tilde{q}_\sigma p_{4\rho}g^{\sigma\rho} = 16m_e^2(\tilde{q}\cdot p_4) \tag{119}$$

For $\tilde{q} = p_1 - p_3$
$$= -8m_e^2 u + 8m_e^4 + 8m_e^2 s \tag{120}$$

For $\tilde{q} = p_1 + p_2$
$$= -8m_e^2 t - 8m_e^2 u + 24m_e^4 \tag{121}$$

3.
$$m_e^2 q_\eta\tilde{q}_\sigma\mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\nu\gamma_\mu\gamma^\sigma\gamma_\nu] = 4m_e^2 q_\eta\tilde{q}_\sigma\mathrm{Tr}[\gamma^\sigma\gamma_\eta] = 16m_e^2 q_\eta\tilde{q}_\sigma g^{\sigma\eta} \tag{122}$$
$$= 16m_e^2(q\cdot\tilde{q}) \tag{123}$$

For $q = p_1 - p_3, \tilde{q} = p_1 + p_2$
$$= 16m_e^2(p_1^2 + (p_1\cdot p_2) - (p_1\cdot p_3) - p_2\cdot p_3) = 8m_e^2(s+t+u) \tag{124}$$

4.

$$q_\eta p_{1\lambda}\tilde{q}_\sigma p_{4\rho}\mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\nu\gamma^\lambda\gamma_\mu\gamma^\sigma\gamma_\nu\gamma^\rho] = -2q_\eta p_{1\lambda}\tilde{q}_\sigma p_{4\rho}\mathrm{Tr}[\gamma^\lambda\gamma^\nu\gamma^\eta\gamma^\sigma\gamma_\nu\gamma^\rho] \tag{125}$$

$$= -8q_\eta p_{1\lambda}\tilde{q}_\sigma p_{4\rho}g^{\eta\sigma}\mathrm{Tr}[\gamma^\lambda\gamma^\rho] = -32q_\eta p_{1\lambda}\tilde{q}_\sigma p_{4\rho}g^{\eta\sigma}g^{\lambda\rho} \tag{126}$$

$$= -32(q\cdot\tilde{q})(p_1\cdot p_4) \tag{127}$$

For $q = p_1 - p_3, \tilde{q} = p_1 + p_2$

$$= -8\left(2m_e^2 - u\right)(s+t+u) \tag{128}$$

5.

$$m_e^2 q_\eta p_{1\lambda}\mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\nu\gamma^\lambda\gamma_\mu\gamma_\nu] = -2m_e^2 q_\eta p_{1\lambda}\mathrm{Tr}[\gamma^\lambda\gamma^\nu\gamma^\eta\gamma_\nu] = 4m_e^2 q_\eta p_{1\lambda}\mathrm{Tr}[\gamma^\lambda\gamma^\eta] \tag{129}$$

$$= 16m_e^2 q_\eta p_{1\lambda}g^{\lambda\eta} = 16m_e^2(q\cdot p_1) \tag{130}$$

6.

$$m_e^2 q_\eta p_{4\rho}\mathrm{Tr}[\gamma^\mu\gamma^\eta\gamma^\nu\gamma_\mu\gamma_\nu\gamma^\rho] = 4m_e^2 q_\eta p_{4\rho}\mathrm{Tr}[\gamma^\eta\gamma^\rho] = 16m_e^2 q_\eta p_{4\rho}g^{\eta\rho} \tag{131}$$

$$= 16m_e^2(q\cdot p_4) \tag{132}$$

7.

$$m_e^2 p_{1\lambda}p_{4\rho}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\mu\gamma_\nu\gamma^\rho] = 4m_e^2 p_{1\lambda}p_{4\rho}\mathrm{Tr}[\gamma^\lambda\gamma^\rho] \tag{133}$$

$$= 16m_e^2 p_{1\lambda}p_{4\rho}g^{\lambda\rho} = 16m_e^2(p_1\cdot p_4) \tag{134}$$

$$= 16m_e^4 - 8m_e^2 u \tag{135}$$

8.

$$m_e^4\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma_\mu\gamma_\nu] = -32m_e^4 \tag{136}$$

Putting it all together,

$$\begin{aligned}
&\frac{1}{4}\mathrm{Tr}[\gamma^\mu(\not{q}+m_e)\gamma^\nu(\not{p}_1+m_e)\gamma_\mu(\not{\tilde{q}}+m_e)\gamma_\nu(\not{p}_4+m_e)] \\
&= \frac{1}{4}\Big[16m_e^2(p_1\cdot\tilde{q}) + 16m_e^2(\tilde{q}\cdot p_4) + 16m_e^2(q\cdot\tilde{q}) \\
&- 32(q\cdot\tilde{q})(p_1\cdot p_4) + 16m_e^2(q\cdot p_1) + 16m_e^2(q\cdot p_4) \\
&+ 16m_e^2(p_1\cdot p_4) - 32m_e^4\Big]
\end{aligned} \tag{137}$$

$$= -8m_e^4+4m_e^2[(p_1\cdot\tilde{q})+(\tilde{q}\cdot p_4)+(q\cdot\tilde{q})+(q\cdot p_1)+(q\cdot p_4)+(p_1\cdot p_4)]-8(q\cdot\tilde{q})(p_1\cdot p_4) \quad (138)$$

Finally we can rewrite the cross-terms in Eq.(23):

$$\begin{aligned}&\frac{1}{ts}[\bar{u}_4\not{\epsilon}_2 Q'\not{\epsilon}_3^* u_1][\bar{u}_1\not{\epsilon}_2^* Q''\not{\epsilon}_3 u_4]\\ &\quad=\frac{1}{ts}\Bigg[-8m_e^4+4m_e^2[p_1\cdot(p_1+p_2)+(p_1-p_3)\cdot(p_1+p_2)+(p_1-p_3)\cdot p_1 \\ &\quad+(p_1-p_3)\cdot p_4+p_1\cdot p_4]-8[(p_1-p_3)\cdot(p_1+p_2)(p_1\cdot p_4)]\Bigg]\end{aligned} \quad (139)$$

$$= \frac{1}{ts}\left[4m_e^2\left(3p_1^2+2p_1\cdot(p_2-p_3+p_4)-p_3\cdot(p_2+p_4)\right)-8m_e^4-8(p_1\cdot p_4)\,(p_1+p_2)\cdot(p_1-p_3)\right] \quad (140)$$

$$= 16m_e^4+16m_e^2 s-8m_e^2 u \quad (141)$$

$$= -8(4m_e^4+m_e^2(s-m_e^2)+m_e^2(t-m_e^2)) \quad (142)$$

Clearly, the last term in Eq.(23) yields the same contribution thus, we can not write out the full averaged-squared matrix element.

$$\begin{aligned}\langle|\mathcal{M}_{\text{tot}}|^2\rangle &= \frac{1}{t^2}\left[16m_e^4+4m_e^2(p_1\cdot p_4)-16m_e^2\,(p_1-p_3)\,(p_3+p_4)+(p_1\cdot p_4)(p_1-p_3)^2\right]\\ &+\frac{1}{s^2}\left[16m_e^4+4m_e^2(p_1\cdot p_4)+16m_e^2(p_1+p_2)(p_2-p_4)+(p_1\cdot p_4)(p_1+p_2)^2\right]\\ &+\frac{2}{ts}\left[4m_e^2\left(3p_1^2+2p_1\cdot(p_2-p_3+p_4)-p_3\cdot(p_2+p_4)\right)-8m_e^4-8(p_1\cdot p_4)\,(p_1+p_2)\cdot(p_1-p_3)\right]\end{aligned} \quad (143)$$

$$\begin{aligned}\langle|\mathcal{M}_{\text{tot}}|^2\rangle &= \frac{1}{t^2}\Bigg[16p_2^2m_e^2+16(p_1\cdot p_2)m_e^2-12(p_1\cdot p_4)m_e^2-16(p_2\cdot p_4)m_e^2+16m_e^4\\ &+p_1^2(p_1\cdot p_4)+p_1p_2^2p_4+2p_1^2p_2p_4\Bigg]\\ &+\frac{1}{s^2}\left[16m_e^4+4m_e^2(p_1\cdot p_4)+16m_e^2(p_1+p_2)(p_2-p_4)+(p_1\cdot p_4)(p_1+p_2)^2\right]\\ &+\frac{2}{ts}\left[4m_e^2\left(3p_1^2+2p_1\cdot(p_2-p_3+p_4)-p_3\cdot(p_2+p_4)\right)-8m_e^4-8(p_1\cdot p_4)\,(p_1+p_2)\cdot(p_1-p_3)\right]\end{aligned} \quad (144)$$

$$= 2e^4\left[\frac{(p_1\cdot p_3)}{(p_1\cdot p_2)}+\frac{(p_1\cdot p_2)}{(p_1\cdot p_3)}+2m_e^2\left(\frac{1}{(p_1\cdot p_2)}-\frac{1}{(p_1\cdot p_3)}\right)+m_e^4\left(\frac{1}{p_1\cdot p_2}-\frac{1}{(p_1\cdot p_3)}\right)^2\right] \quad (145)$$

1.3 Kinematics

To further simplify we need some kinematics. We align our axes so that the photon is travelling along the z-axis. In the lab frame we have

$$p_1 = (m_e, 0, 0, 0), \qquad p_4 = (E, -\mathbf{p}_3) \tag{146}$$

$$p_2 = (\omega, 0, 0, \omega), \qquad p_3 = (\omega', \omega' \sin\theta, 0, \omega' \cos\theta) \tag{147}$$

We have the Mandelstam variables

$$\begin{aligned} t &= (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3 = m_e^2 - 2m_e\omega' \\ &= (p_4 - p_2)^2 = p_4^2 + p_2^2 - 2p_4 \cdot p_2 = E^2 - \omega'^2 - 2E\omega - 2\omega\omega' \cos\theta \end{aligned} \tag{148}$$

$$\begin{aligned} (p_1 \cdot p_3) &= -\frac{t}{2} + \frac{m_e^2}{2} = m_e\omega' \\ (p_4 \cdot p_2) &= -\frac{t}{2} + \frac{E^2 - \omega'^2}{2} = E\omega + \omega\omega' \cos\theta \end{aligned} \tag{149}$$

$$\begin{aligned} s &= (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = m_e^2 + 2m_e\omega \\ &= (p_3 + p_4)^2 = p_3^2 + p_4^2 + 2p_3 \cdot p_4 = E^2 - \omega'^2 + 2E\omega' + 2\omega'^2 \end{aligned} \tag{150}$$

$$\begin{aligned} (p_1 \cdot p_2) &= \frac{s}{2} - \frac{m_e^2}{2} = m_e\omega \\ (p_3 \cdot p_4) &= \frac{s}{2} - \frac{E^2 - \omega'^2}{2} = E\omega' + \omega'^2 \end{aligned} \tag{151}$$

$$\begin{aligned} u &= (p_1 - p_4)^2 = p_1^2 + p_4^2 - 2p_1 \cdot p_4 = m_e^2 + E^2 - \omega'^2 - 2m_eE \\ &= (p_3 - p_2)^2 = p_3^2 + p_2^2 - 2p_3 \cdot p_2 = -2\omega\omega' + 2\omega\omega' \cos\theta \end{aligned} \tag{152}$$

$$\begin{aligned} (p_1 \cdot p_4) &= -\frac{u}{2} + \frac{m_e^2}{2} + \frac{E^2 - \omega'^2}{2} = m_eE \\ (p_3 \cdot p_2) &= -\frac{u}{2} = \omega\omega'(1 - \cos\theta) \end{aligned} \tag{153}$$

We must also note

$$m_e^2 = p_4^2 = (p_1 + p_2 - p_3)^2 = m_e^2 + 2m_e(\omega - \omega') - 2\omega\omega'(1 - \cos\theta) \tag{154}$$

Thus,

$$\frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m_e}(1 - \cos\theta) \rightsquigarrow \omega' = \frac{\omega}{1 + \frac{\omega}{m_e}(q - \cos\theta)} \tag{155}$$

In the lab frame, the phase space integral yields

$$\int \frac{d^3\mathbf{p_3}}{(2\pi)^3}\frac{1}{2\omega'}\frac{d^3\mathbf{p_4}}{(2\pi)^3}\frac{1}{2E}(2\pi)^4\delta^4(p_3+p_4-p_1-p_2) \tag{156}$$

$$= \frac{1}{8\pi}\int d\cos\theta\,\frac{\omega'^2}{\omega m_e} \tag{157}$$

Thus, the differential cross section is given by

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2\omega}\frac{1}{2m}\frac{1}{8\pi}\frac{\omega'^2}{\omega m_e}\left[\frac{1}{4}\langle|\mathcal{M}|^2\rangle\right] \tag{158}$$

Plugging everything in and simplifying yields the following results

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m_e^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega}+\frac{\omega}{\omega'}-\sin^2\theta\right] \tag{159}$$

also known as the *Klein-Nishima formula*.