BHABHA SCATTERING

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1 Bhabha Scattering

Here we compute the differential cross section $d\sigma/d\Omega$ for Bhabha scattering ($e^+e^- \to e^+e^-$) in the center-of-mass frame.

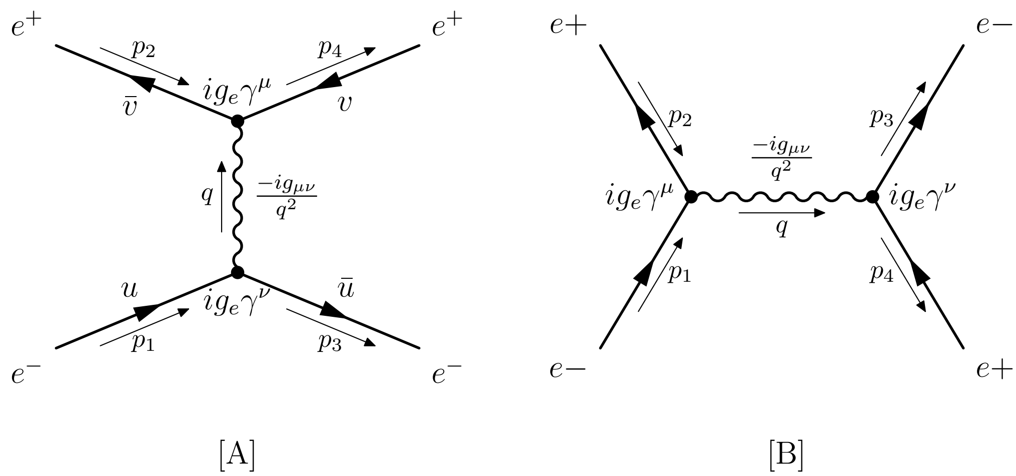

Figure 1: [A] Shows the t-channel Feynman diagram and [B] shows the s-channel Feynman diagram

The corresponding matrix elements are given by

$$\mathcal{M}_t = \int \frac{d^4q}{(2\pi)^4}[\bar{v}(\mathbf{p_2},\sigma_2)ig_e\gamma^\mu v(\mathbf{p}_4,\sigma_4)]\frac{-ig_{\mu\nu}}{q^2}[\bar{u}(\mathbf{p}_3,\sigma_3)ig_e\gamma^\nu u(\mathbf{p}_1,\sigma_1)](2\pi)^4\delta^4(p_1-q-p_3) \tag{1}$$

$$= \frac{-g_e^2}{(p_1-p_3)^2}[\bar{v}(\mathbf{p_2},\sigma_2)\gamma^\mu v(\mathbf{p}_4,\sigma_4)][\bar{u}(\mathbf{p}_3,\sigma_3)\gamma_\mu u(\mathbf{p}_1,\sigma_1)] \tag{2}$$

$$\mathcal{M}_s = \int \frac{d^4q}{(2\pi)^4}[\bar{u}(\mathbf{p_3},\sigma_3)ig_e\gamma^\mu v(\mathbf{p}_4,\sigma_4)]\frac{-ig_{\mu\nu}}{q^2}[\bar{v}(\mathbf{p}_2,\sigma_2)ig_e\gamma^\nu u(\mathbf{p}_1,\sigma_1)](2\pi)^4\delta^4(p_1+p_2-q) \tag{3}$$

$$= \frac{-g_e^2}{(p_1+p_2)^2}[\bar{u}(\mathbf{p_3},\sigma_3)\gamma^\mu v(\mathbf{p}_4,\sigma_4)][\bar{v}(\mathbf{p}_2,\sigma_2)\gamma_\mu u(\mathbf{p}_1,\sigma_1)] \tag{4}$$

The total amplitude is then given by

$$\mathcal{M}_{\text{tot}} = \mathcal{M}_t - \mathcal{M}_s \tag{5}$$

And thus,

$$|\mathcal{M}_{\text{tot}}| = |\mathcal{M}_t|^2 + |\mathcal{M}_s|^2 - \mathcal{M}_t\mathcal{M}_s^\dagger - \mathcal{M}_s\mathcal{M}_t^\dagger \tag{6}$$

We also want to average over initial spins and sum over final spins. Making a straightforward switch of notation and computing each of the terms in Eq.(6) separately

$$\langle|\mathcal{M}_t|^2\rangle = \frac{g_e^4}{t^2}\frac{1}{4}\sum_{\sigma_i=1,2}[\bar{v}_2\gamma^\mu v_4][\bar{u}_3\gamma_\mu u_1][\bar{v}_4\gamma^\nu v_2][\bar{u}_1\gamma_\nu u_3] \qquad \text{for } i=1,2,3,4 \tag{7}$$

$$= \frac{g_e^4}{4t^2}\sum_{i,j,l,m=1}^{4}[\gamma^\mu(\not{p}_4 - m_e c)\gamma^\nu]_{ij}[\gamma_\mu(\not{p}_1 + m_e c)\gamma_\nu]_{lm}\sum_{\sigma_2=1,2}[\bar{u}_2 u_2]_{ji}\sum_{\sigma_3=1,2}[\bar{u}_3 u_3]_{ml} \tag{8}$$

$$= \frac{g_e^4}{4t^2}\sum_{i,l=1}^{4}[\gamma^\mu(\not{p}_4 + m_e c)\gamma^\nu(\not{p}_2 + m_e c]_{ii}[\gamma_\mu(\not{p}_1 + m_e c)\gamma_\nu(\not{p}_3 + m_e c)]_{ll} \tag{9}$$

$$= \frac{g_e^4}{4t^2}\text{Tr}[\gamma^\mu(\not{p}_4 + m_e c)\gamma^\nu(\not{p}_2 - m_e c)]\text{Tr}[\gamma_\mu(\not{p}_1 + m_e c)\gamma_\nu(\not{p}_3 + m_e c)] \tag{10}$$

In the limit where $\mathbf{p}_i \gg m_e$ we can neglect the mass of the electron. Using the identity $\text{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})$ and doing a very similar calculation to the one we did in problem set three (see Eqs.[126]-[130] in PS3) we find

$$= \frac{8g_e^4}{t^2}[(p_1\cdot p_2)(p_4\cdot p_3) + (p_4\cdot p_1)(p_2\cdot p_3)] \tag{11}$$

The second term in Eq.(6) gives

$$\langle|\mathcal{M}_s|^2\rangle = \frac{g_e^4}{s^2}\frac{1}{4}\sum_{\sigma_i=1,2}[\bar{u}_3\gamma^\mu v_4][\bar{v}_2\gamma_\mu u_1][\bar{v}_4\gamma^\nu u_3][\bar{u}_1\gamma_\nu v_2] \qquad \text{for } i=1,2,3,4 \tag{12}$$

$$= \frac{8g_e^4}{s^2}[(p_1\cdot p_3)(p_4\cdot p_2) + (p_4\cdot p_1)(p_3\cdot p_2)] \tag{13}$$

Now for the last two terms in Eq.(6),

$$\langle\mathcal{M}_t\mathcal{M}_s^*\rangle = \frac{g_e^4}{st}\frac{1}{4}\sum_{\sigma_i=1,2}[\bar{v}_2\gamma^\mu v_4][\bar{u}_3\gamma_\mu u_1][\bar{v}_4\gamma^\nu u_3][\bar{u}_1\gamma_\nu v_2] \tag{14}$$

$$= \frac{g_e^4}{4st}\text{Tr}[\gamma^\mu(\not{p}_4 - m_e c)\gamma^\nu(\not{p}_3 + m_e c)\gamma_\mu(\not{p}_1 + m_e c)\gamma_\nu(\not{p}_2 - m_e c)] \tag{15}$$

Again, following problem set three (Eqs.[141]-[145])

$$= \frac{-8g_e^4}{st}(p_3 \cdot p_2)(p_4 \cdot p_1) \tag{16}$$

Likewise,

$$\langle \mathcal{M}_s \mathcal{M}_t^* \rangle = \frac{g_e^4}{st}\frac{1}{4}\sum_{\sigma_i=1,2}[\bar{u}_3\gamma^\mu v_4][\bar{v}_2\gamma_\mu u_1][\bar{v}_4\gamma^\nu v_2][\bar{u}_1\gamma_\nu u_3] \tag{17}$$

$$= \frac{-8g_e^4}{st}(p_3 \cdot p_2)(p_4 \cdot p_1) \tag{18}$$

Putting it all together

$$\begin{aligned}\langle|\mathcal{M}_{\text{tot}}|^2\rangle = 8g_e^4\Bigg[&\frac{1}{t^2}\left[(p_1 \cdot p_2)(p_4 \cdot p_3) + (p_4 \cdot p_1)(p_2 \cdot p_3)\right] \\ &+ \frac{1}{s^2}\left[(p_1 \cdot p_3)(p_4 \cdot p_2) + (p_4 \cdot p_1)(p_3 \cdot p_2)\right] + \frac{2}{st}(p_3 \cdot p_2)(p_4 \cdot p_1)\Bigg]\end{aligned} \tag{19}$$

If the electron/positron's mass is neglected we have

$$t = (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3 = -2p_1 \cdot p_3 \tag{20}$$

$$s = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = 2p_1 \cdot p_2 \tag{21}$$

From energy-momentum conservation $p_1 + p_2 = p_3 + p_4$

$$(p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = 2p_1 \cdot p_2 \tag{22}$$

$$(p_3 + p_4)^2 = p_3^2 + p_4^2 + 2p_3 \cdot p_4 = 2p_3 \cdot p_4 \tag{23}$$

Implying that $p_1 \cdot p_2 = p_3 \cdot p_4$. It also implies $p_1 \cdot p_3 = p_2 \cdot p_4$ and $p_1 \cdot p_4 = p_2 \cdot p_3$. This gives

$$\begin{aligned}\langle|\mathcal{M}_{\text{tot}}|^2\rangle = 8g_e^4\Bigg[&\frac{1}{4(p_1 \cdot p_3)^2}\left[(p_1 \cdot p_2)^2 + (p_1 \cdot p_4)^2\right] \\ &+ \frac{1}{4(p_1 \cdot p_2)^2}\left[(p_1 \cdot p_3)^2 + (p_1 \cdot p_4)^2\right] + \frac{2}{4(p_1 \cdot p_3)(p_1 \cdot p_2)}(p_1 \cdot p_4)^2\Bigg]\end{aligned} \tag{24}$$

$$= 2g_e^4\left[\frac{(p_1 \cdot p_2)^2}{(p_1 \cdot p_3)^2} + \frac{(p_1 \cdot p_3)^2}{(p_1 \cdot p_2)^2} + (p_1 \cdot p_4)^2\left[\frac{[(p_1 \cdot p_2) + (p_1 \cdot p_3)]^2}{(p_1 \cdot p_2)^2(p_1 \cdot p_3)^2}\right]\right] \tag{25}$$

The last term can be rewritten as

$$\big(p_1 \cdot (p_2 + p_3)\big)^2 = \big(p_1 \cdot (p_1 + p_4)\big)^2 = \big(p_1^2 + (p_1 \cdot p_4)\big)^2 = (p_1 \cdot p_4)^2 \tag{26}$$

$$= \frac{2g_e^4}{(p_1 \cdot p_2)^2 (p_1 \cdot p_3)^2} \left[(p_1 \cdot p_2)^4 + (p_1 \cdot p_3)^4 + (p_1 \cdot p_4)^4 \right] \tag{27}$$

In the center-of-mass frame

$$\mathbf{p}_1 = -\mathbf{p}_2, \qquad \mathbf{p}_3 = -\mathbf{p}_4, \qquad \mathbf{p}_1 \cdot \mathbf{p}_3 = \frac{E^2}{c^2} \cos\theta \tag{28}$$

$$\frac{E^2}{c^2} = \mathbf{p}_1^2 = \mathbf{p}_2^2 = \mathbf{p}_3^2 = \mathbf{p}_4^2 \tag{29}$$

$$p_1 \cdot p_2 = \frac{E^2}{c^2} + \mathbf{p}_1 \cdot \mathbf{p}_2 = 2\frac{E^2}{c^2} \tag{30}$$

$$p_1 \cdot p_3 = \frac{E^2}{c^2} - \mathbf{p_1} \cdot \mathbf{p}_3 = \frac{E^2}{c^2}(1 - \cos\theta) \tag{31}$$

$$p_1 \cdot p_4 = \frac{E^2}{c^2} + \mathbf{p}_1 \cdot \mathbf{p}_4 = \frac{E^2}{c^2}(1 + \cos\theta) \tag{32}$$

Plugging back into Eq.[27]

$$= \frac{2g_e^4 c^8}{E^8 (1-\cos\theta)^2 (1+\cos\theta)^2} \frac{E^8}{c^8} \left[16 + (1-\cos\theta)^4 + (1-\cos\theta)^4 \right] \tag{33}$$

$$\langle |\mathcal{M}_{\mathrm{tot}}|^2 \rangle = 2g_e^4 \left[\frac{(\cos(2\theta)+7)^2}{2\sin^4\theta} \right] = \left[\frac{g_e^2(\cos(2\theta)+7)}{\sin^2\theta} \right]^2 \tag{34}$$

The differential cross section in the center-of-mass frame is given by (see problem set two Eq.[54])

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi} \right)^2 \frac{\langle |\mathcal{M}|^2 \rangle}{(E_a + E_b)^2} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|} \tag{35}$$

Thus,

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{16\pi E} \left[\frac{g_e^2(\cos(2\theta)+7)}{\sin^2\theta} \right] \right)^2 \tag{36}$$

For electron-muon scattering the calculation is essentially the same with on an s-channel diagram to analyze, thus, I quote my derived result in the limit where the masses of the muon and electron are ignored

$$\frac{d\sigma}{d\Omega} = \left[\left(\frac{\hbar c g_e^2}{8\pi E} \right)^2 \frac{1}{2} \left(\frac{1 + \cos^4\theta/2}{\sin^4\theta/2} \right) \right] \tag{37}$$

They are not the same!