LORENTZ INVARIANT TWO-BODY PHASE SPACE

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1 2-body phase space

We want to prove the result that the outgoing 2-particle Lorentz invariant phase space is equal to the following

$$d\text{Lips}(s; P_1', P_2') = \frac{p d\Omega}{16\pi^2 \sqrt{s}} \tag{1}$$

This so-called Lorentz invariant phase space is important when we are interested in calculating cross-sections or decay rates for two body collisions at relativistic speeds (like that of a particle collider). The phase space governs the potential trajectories of a system and tells us something about the density of states, ultimately it tells us which states can be reached by the system.

First we need to define the kinematics of the problem. We imagine two particles speeding towards each other and colliding. We define the following outgoing 4-momentum

$$\begin{aligned} P_1' &= (E', \mathbf{p_1'}) \\ P_2' &= (E', \mathbf{p_2'}) \end{aligned} \tag{2}$$

$$P \equiv (P_1' + P_2') \tag{3}$$

$$P^2 = (P_1' + P_2')^2 \equiv s \tag{4}$$

In the center-of-mass frame we have

$$\begin{aligned} \mathbf{p_1'} &= \mathbf{p} = -\mathbf{p_2'} \\ s &= (E_1 + E_2)^2 \end{aligned} \tag{5}$$

$$p \equiv |\mathbf{p}'| \tag{6}$$

$$E_{CM} \equiv \sqrt{s} \tag{7}$$

Thus,

$$s = (P_1' + P_2')^2 = m_1^2 + m_2^2 + 2P_1 \cdot P_2 \tag{8}$$

$$P_1 \cdot P_2 = E_1 E_2 - \mathbf{p_1} \cdot \mathbf{p_2} = E_1 E_2 + \mathbf{p}^2 \tag{9}$$

Plugging back into (7)

$$s = m_1^2 + m_2^2 + 2(E_1 E_2 + p^2) \tag{10}$$

$$(s - m_1^2 - m_2^2) = 2(E_1 E_2 + p^2) \tag{11}$$

Remembering that $E_1 = \sqrt{m_1^2 + p^2}$ and $E_2 = \sqrt{m_2^2 + p^2}$ we find that the following relation is true

$$p\sqrt{s} = \frac{1}{2}\sqrt{\lambda} \tag{12}$$

Where λ is

$$\begin{aligned}\lambda(s, m_1, m_2) &\equiv [s-(m_1+m_2)^2][s-(m_1-m_2)^2] \\ &= m_1^4 - 2m_1^2 m_2^2 + m_2^4 - 2m_1^2 s - 2m_2^2 s + s^2\end{aligned} \tag{13}$$

Now that we have all of the necessary kinematic information let's begin the calculation. We start with the definition of the infinitesimal Lorentz invariant phase space (dLips) of a system with center of mass energy s, and momentum $P_1, P_2, ..., P_n$

$$d\text{Lips}(s; P_1, \cdots, P_n) = (2\pi)^4 \delta^4(p_i - p_f) d\text{Lips}(P_1, \cdots, P_n) \tag{14}$$

For two-body phase space we have

$$d\text{Lips}(s; P_1', P_2') = (2\pi)^4 \delta^4(P - P_1' + P_2') \frac{1}{(2\pi)^6} \frac{d^3\mathbf{p}_1'}{2E_1'} \frac{d^3\mathbf{p}_2'}{2E_2'} \tag{15}$$

Now, we can split up the four dimensional delta function into two delta functions. One which ensures conservation of momentum and the other which ensures conservation of energy.

$$= \frac{1}{16\pi^2} \delta^3(\mathbf{P} - \mathbf{p_1'} + \mathbf{p_2'}) \delta(\sqrt{s} - E_1' + E_2') \frac{d^3\mathbf{p}_1'}{E_1'} \frac{d^3\mathbf{p}_2'}{E_2'} \tag{16}$$

In the center of mass frame $\mathbf{P} = 0$

$$= \frac{1}{16\pi^2} \delta^3(\mathbf{p_1'} + \mathbf{p_2'}) \delta(\sqrt{s} - E_1' + E_2') \frac{d^3\mathbf{p}_1'}{E_1'} \frac{d^3\mathbf{p}_2'}{E_2'} \tag{17}$$

In the center of mass frame we can now integrate over $\mathbf{p}_2'$ in the sense that we are integrating over an arbitrary function of $\mathbf{p}_2'$

$$\int \delta^3(\mathbf{p_1'} + \mathbf{p_2'}) f(\mathbf{p'}_1) d^3\mathbf{p}_2' = f(-\mathbf{p}_1') \tag{18}$$

We just end up with an arbitrary function of $\mathbf{p}_1'$. After substituting (5) we are left with

$$= \frac{1}{16\pi^2} \delta(\sqrt{s} - E_1' + E_2') \frac{d^3\mathbf{p}}{E_1' E_2'} \tag{19}$$

Currently we are in cartesian coordinates, it will be more convenient to be in spherical coordinates.

$$d^3\mathbf{p} = d\mathbf{p}_x d\mathbf{p}_y d\mathbf{p}_z \tag{20}$$

We can use the determinant of the Jacobian matrix to transform from cartesian coordinates (x, y, z) to spherical coordinates (r, θ, ϕ)

$$\mathbf{J}_{ij} = \frac{\partial f_i}{\partial x_j} \tag{21}$$

We remember that

$$\begin{aligned} x(r,\theta,\phi) &= r\sin(\theta)\cos(\phi) \\ y(r,\theta,\phi) &= r\sin(\theta)\sin(\phi) \\ z(r,\theta,\phi) &= r\cos(\theta) \end{aligned} \tag{22}$$

The Jacobian matrix is the following

$$\mathbf{J}_{ij} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \cos(\phi)\cos(\theta) & r\cos(\theta)\cos(\phi) & -r\sin(\phi)\sin(\theta) \\ \sin(\theta)\sin(\phi) & r\cos(\theta)\sin(\phi) & r\cos(\phi)\cos(\theta) \\ \cos(\theta) & -r\sin(\theta) & 0 \end{pmatrix} \tag{23}$$

The determinant of this matrix will tell us the appropriate factor of "stretching" or rotating which the transformation causes locally. The determinant of the above matrix is equal to $r^2\sin(\theta)$. A volume element in spherical coordinates will take the form of

$$dV = r^2\sin(\theta)drd\theta d\phi \tag{24}$$

For a volume element in phase space we say

$$dV_{ps} = |\mathbf{p}|^2\sin(\theta)d|\mathbf{p}|d\theta d\phi = p^2dpd\Omega \tag{25}$$

Where $sin(\theta)d\theta d\phi = d\Omega$ is the differential solid angle. Plugging this result into (20)

$$= \frac{1}{16\pi^2}\delta(\sqrt{s} - E_1' + E_2')\frac{p^2dpd\Omega}{E_1'E_2'} \tag{26}$$

We can rewrite the differential momentum in the following way

$$E = E_1' + E_2' = \sqrt{m_1^2 + \mathbf{p}^2} + \sqrt{m_2^2 + \mathbf{p}^2} \tag{27}$$

$$\frac{dE}{dp} = \frac{p}{\sqrt{m_1^2 + \mathbf{p}^2}} + \frac{p}{\sqrt{m_2^2 + \mathbf{p}^2}} = \frac{p}{E_1'} + \frac{p}{E_2'} = \frac{p(E_1' + E_2')}{E_1'E_2'} \tag{28}$$

$$dp = \frac{E_1'E_2'dE}{pE} \tag{29}$$

Plugging back into (27) we get

$$= \frac{1}{16\pi^2}\delta(\sqrt{s} - E)\frac{pdEd\Omega}{E} \tag{30}$$

After integrating over E as we did in (19) we are left with the expected result

$$d\text{Lips}(s; P_1', P_2') = \frac{1}{16\pi^2}\frac{pd\Omega}{E} \tag{31}$$