HOLOGRAPHIC DICTIONARY

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1 String theory necessities

In the study of two-dimensional objects propagating in d dimensional space there are two important string classes to investigate: open and closed strings.

1.1 Open strings

The bosonic world-sheet action for a two-dimensional object (string) propagating in a d-dimensional space is given by: Insert bosonic string action from holographic_TC.tex

Varying the action we obtain the following equations of motion and boundary conditions:

$$\partial^\alpha \partial_\alpha X^\mu = 0 \tag{1}$$

$$\partial_\sigma X \cdot \delta X \Big|_{\sigma=\ell} - \partial_\sigma \cdot \delta X \Big|_{\sigma=0} = 0 \tag{2}$$

The boundary condition admits two solutions...

To get rid of tachyonic states $m < 0$ we classify states based on there 'GSO-projection' and only keep those states which satisfy ...

1.2 Type II theories

Here we focus on closed superstring theories. As shown above we have four sectors which we must perform GSO projections. By analyzing the massless part of the spectrum of states we find that NS-NS sector of type IIB theories admit the following bosonic massless spectrum (those states which satisfy the massless GSO projection condition): a graviton $h_{\mu\nu}$, a two-form gauge potential $B_{(2)}$, and a scalar field Φ.

1.3 String theory → Gauge theory

The gauge/gravity correspondence stems from the duality between open and closed strings. The low energy dynamics of massless open strings on a Dp-brane gives rise to a $(p+1)$-dimensional SUSY gauge theory while a Dp-brane in the closed string perspective is described as a classical solution of the low-energy supergravity equations of motion charged under the R-R field C_{p+1} (boundary of the conformal field theory of closed strings). This is the basis of the gauge/gravity correspondence which we will now clarify. First, we find a background solution to a theory of supergravity. This includes finding solutions to the metric and any other p-forms which the theory may be charged under. Once the solution is obtained we can then plug this solution back into the original action, expand with a special truncation and obtain information about the dual gauge theory. In particular, we will be interested in finding the properties of the dual gauge coupling constant as well as the non-perturbative θ-angle. The truncation can be viewed as taking the leading order terms in the $\ell_s \to 0$ limit. First, we split the coordinates into two groups: those parallel and perpendicular to the world volume.

Consider N D-branes in flat Minkowski space.

2 String theory

We will begin by discussing the motion of p-dimensional objects or p-branes moving through D-dimensional (d space dimensions + 1 time dimension) spacetime ($D \geq p+1$). We will derive the equations of motion for a few relevant systems.

2.1 Point particle (0-brane)

We'll start with the most familiar system, i.e. a massive relativistic point particle moving through four-dimensional spacetime. The action must be Lorentz scalar, with dimension $[S] = 0$. The simplest choice which satisfies both of these constraints would be proportional to the proper distance traversed over some path $\mathcal{P}$

$$S = -m \int_{\mathcal{P}} ds \tag{3}$$

where m is some constant with dimensions of mass ($[ds] = E^{+1}$) naturally associated with the mass of the point particle and ds is an infinitesimal element of the proper distance

$$ds = \sqrt{g_{\mu\nu} dx^\mu dx^\nu} = dt \sqrt{1 - \left(\frac{d\vec{x}}{dt}\right)^2} \tag{4}$$

Variation of the action gives

$$\delta S = 0 = -m \int \delta(ds) \tag{5}$$

To proceed further we can parameterize the path of the particle

$$x^\mu = x^\mu(\tau) \tag{6}$$

giving

$$ds = \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}d\tau^2} \tag{7}$$

the variation is then given by

$$\delta(ds) = \frac{1}{2}ds^{-1}\left(g_{\mu\nu}\delta\left(\frac{dx^\mu}{d\tau}\right)\frac{dx^\nu}{d\tau}d\tau^2 + g_{\mu\nu}\frac{dx^\mu}{d\tau}\delta\left(\frac{dx^\nu}{d\tau}\right)d\tau^2\right) \tag{8}$$

$$= ds^{-1}g_{\mu\nu}\delta\left(\frac{dx^\mu}{d\tau}\right)\frac{dx^\nu}{d\tau}d\tau^2 \tag{9}$$

$$= \frac{d(\delta x^\mu)}{d\tau}\frac{dx_\mu}{ds}d\tau \tag{10}$$

Integrating by parts

$$\delta S = -m\delta x^\mu(\tau)\frac{dx_\mu(\tau)}{ds}\Bigg|_{\tau_i}^{\tau_f} + \int_{\tau_i}^{\tau_f}\left[\frac{d}{d\tau}\left(m\frac{dx^\mu}{ds}\right)\right]\delta x_\mu d\tau \tag{11}$$

The variation of the path coordinates at the endpoints is fixed i.e. $\delta x_\mu(\tau_i) = \delta x_\mu(\tau_f) = 0$. Thus, the equations of motion are given by

$$\frac{d}{d\tau}\left(m\frac{dx_\mu}{ds}\right) = 0 \tag{12}$$

Of course, we recognize that $m(dx_\mu/ds)$ is a four-momentum p_μ and write

$$\dot{p}_\mu = 0 \tag{13}$$

Note that nowhere in our derivation did we rely on a specific number of dimensions in which our point-particle propagated through. The above holds for any D-dimensional space in which Lorentz invariance holds.

Defining $dx_\mu/d\tau \equiv \dot{x}_\mu$ the action can also be written as

$$S = \int_{\tau_i}^{\tau_f} L d\tau, \qquad L = -m\sqrt{-\dot{x}^2} \tag{14}$$

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}} \tag{15}$$

2.2 String (1-branes)

Strings or equivalently 1-branes sweep out a two-dimensional surface called a *world-sheet*. The action of the string, just as the action of a point particle was proportional to its proper length, will be proportional to the area of it's world-sheet.

$$S = T \int dA \tag{16}$$

where T is some constant with $[T] = E^{+2}$ and termed the tension of the string. The tension is related to the Regge slope

$$T = \frac{1}{2\pi\alpha'}, \quad [\alpha'] = M^{-2} \tag{17}$$

The string length l_s and mass M_s are defined as

$$l_s = \sqrt{\alpha'} = \frac{1}{M_s} \tag{18}$$

Because the string sweeps out a two-dimensional surface and we will require two parameters (τ, σ) to fully parameterize the surface. A given point in the parameter space (τ, σ) is mapped to a $D = d + 1$-dimensional point in spacetime via the string mapping functions $X(\tau, \sigma)$. Under this parameterization, τ roughly correpsonds to time and σ is related to the position on the worldsheet. Generally τ ranges over an infinite interval and σ on a finite interval.

Now we need an expression resembling the area of the world sheet in terms of our string coordinates. We can achieve this by considering the infinitesimal area $d\tau d\sigma$. In spacetime this is a quadrilateral spanned by the two vectors

$$du_1^\mu = \frac{\partial X^\mu}{\partial \tau} d\tau, \qquad du_2^\mu = \frac{\partial X^\mu}{\partial \sigma} d\sigma \tag{19}$$

Using the equation for the area of a parallelogram and defining

$$\dot{X}_\mu \equiv \frac{\partial X_\mu}{\partial \tau}, \qquad X'_\mu \equiv \frac{\partial X_\mu}{\partial \sigma} \tag{20}$$

we find that the area of the world sheet is given by

$$A = \int d\tau d\sigma \sqrt{(\dot{X}^\mu X'_\mu)^2 - \dot{X}^\mu \dot{X}_\mu X^{\mu\prime} X'_\mu} \tag{21}$$

Our string action must be proportional to this area and along with the dimensionless constraint we obtain the *Nambu-Goto* action

$$S = -T \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_0} d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2} \tag{22}$$

By defining the *induced* metric $h_{\alpha\beta}$ on the world-sheet

$$h_{\alpha\beta} \equiv g_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta} \tag{23}$$

where $\xi^\alpha \equiv (\tau, \sigma)$ with $\alpha, \beta = 1, 2$. More explicitly the induced metric can be written as

$$\mathbf{h} = \begin{pmatrix} (\dot{X})^2 & \dot{X} \cdot X' \\ \dot{X} \cdot X' & (X')^2 \end{pmatrix} \tag{24}$$

We can now rewrite the Nambu-Goto action as

$$S = -T \int d\tau d\sigma \sqrt{-h}, \qquad h \equiv \det(h_{\alpha\beta}) \tag{25}$$

which is manifestly reparameterization invariant.

2.3 *p*-brane

A p-brane will sweep out a $(p+1)$-dimensional surface. It's action, just as we have seen before, will be proportional to the world-volume swept out by it's motion

$$S = -T_p \int dV \tag{26}$$

where $[T_p] = E^{+p}$ and is related to the 'tension' of the brane. We will need $p+1$ parameters to fully parameterize the world-volume and choose ξ^a for $a = 0, \ldots, p$. Thus we can write

$$S = -T_p \int d^{p+1}\xi \sqrt{-h} \tag{27}$$

where h is the induced metric (pullback) on the world-volume

$$h_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} g_{\mu\nu} \tag{28}$$

The action in Eq.(27) is referred to as the *Dirac action* who wrote it down originally in the context of membrane dynamics.

2.4 DBI-action

See Imeroni for action expressions and Tong for pedagogical details.
The world-volume action a Dp-brane is given by, in the string frame, as:

$$S_{Dp} = -T_p \int_{\mathcal{M}_{p+1}} d^{p+1}\xi \; e^{-\Phi} \sqrt{-\det\left(\hat{G}_{ab} + \hat{B}_{ab} + 2\pi l_s^2 F_{ab}\right)} + \mu_p \int_{\mathcal{M}_{p+1}} \sum_q \hat{C}_q \wedge e^{\hat{B}+2\pi l_s^2 F} \tag{29}$$

where $\mathcal{M}_{p+1}$ denotes the full Dp-brane world-volume, ξ^a, $a = 0, \ldots, p$ parameterize the world-volume directions, T_p is the brane tension, μ_p is the R-R charge of the D-brane, $\hat{G}_{ab}$, $\hat{B}_{ab}$, and $\hat{C}_{ab}$ are the 'pull-backs' (induced) of the space-time metric, two-form gauge potential of the NS-NS sector, and p-form potentials of the R-R sector onto the world volume of the brane i.e.

$$\hat{G}_{ab} = \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b} G_{\mu\nu} \tag{30}$$

and likewise for B and C. The sum in the second terms is meant to pick up the terms in the expansion corresponding to $(p+1)$-forms, which give a non-vanishing result to the integral over the world-volume. Equivalently in the Einstein frame we have

$$S_{Dp} = -T_p \int_{\mathcal{M}_{p+1}} d^{p+1}\xi \; e^{(p-3)\Phi/4} \sqrt{-\det\left(\hat{G}_{ab} + \hat{B}_{ab} + 2\pi l_s^2 F_{ab}\right)} + \mu_p \int_{\mathcal{M}_{p+1}} \sum_q \hat{C}_q \wedge e^{\hat{B}+2\pi l_s^2 F} \tag{31}$$

Now we must determine the coefficients T_p and μ_p appearing in the world-volume action in terms of the string quantities g_s and l_s. To make this specification requires a comparison between the string vacuum amplitude between two D-branes and the field-theoretic-equivalent amplitude of the exchange of a graviton, dilaton, and R-R field between two D-branes. Given the normalization of the world-volume action in Eq.(31) we find

$$\mu_p = T_p \tag{32}$$

and

$$T_p = \frac{\sqrt{\pi}(2\pi l_s)^{3-p}}{\kappa} = \frac{1}{(2\pi)^p g_s l_s^{p+1}} \tag{33}$$

and thus we have in the string-frame

$$S_{Dp} = -T_p \left\{ \int_{\mathcal{M}_{p+1}} d^{p+1}\xi \; \left[e^{-\Phi} \sqrt{-\det\left(\hat{G}_{ab} + \hat{B}_{ab} + 2\pi l_s^2 F_{ab}\right)} - \sum_q \hat{C}_q \wedge e^{\hat{B}+2\pi l_s^2 F} \right] \right\} \tag{34}$$

3 Branes and String dualities

The d-brane actions contains two peices, the first being the Dirac-Born-Infeld (DBI) action

$$S_{\mathrm{DBI}} = -\mu_p \int_{W_{p+1}} d^{p+1}x e^{-\phi}\sqrt{-\det(P(G+B) - 2\pi\alpha' F)} \tag{35}$$

where the coefficient μ_p is given by

$$\mu_p = \frac{(\alpha')^{-(p+1)/2}}{(2\pi)^p}. \tag{36}$$

This action describes the interations between two parallel d-branes by the exchange of NSNS and RR closed strings. The pullback of a generic tensor A, $P[A]$ into the brane worldvolume, i.e., the induced worldvolume tensor is given by

$$P[G]_{\mu\nu} = G_{\mu\nu} + G_{\mu i}\partial_\nu \phi^i + \partial_\mu \phi^i G_{i\nu} + \partial_\mu \phi^i \partial_\nu \phi^j G_{ij} \tag{37}$$

This introduces the dependence of the action on the embedding fields ϕ^i. The second peice of the effective action describes topological couplings to the RR fields, known as Chern-Simons (CS) terms. They are given by

$$S_{\mathrm{CS}} = \mu_p \int_{W_{p+1}} P\left[\sum_q C_q\right] \wedge e^{2\pi\alpha' F - B_2} \wedge \hat{A}(R) \tag{38}$$

where it is assumed that the action only picks up the terms corresponding to $(p+1)$-forms which may be integrated of the worldvolume W_{p+1}. The first factor corresponds to a formal sum of the spacetime RR q-form fields C_q pulled back on the brane volume. The thirds factor is the A-roof polynomial

$$\hat{A}(R) = 1 - \frac{1}{24(8\pi^2)}\mathrm{Tr}R^2 + \cdots \tag{39}$$

where R is the curvature 2-form defined by [TM: See sect. B.3 in Uranga, Ibanez] and hence, only relevant in the presence of spacetime curvature.

We now quote the supergravity solution for certain p-brane states in the different string theories, focusing on the simplest states, carrying charge under a single p-form field. For any $(p+1)$-form gauge potential, there exists BPS p-brane solutions charged under it. These charges can be measured by computing the H_{8-p} flux around a $(8-p)$-sphere surrounding the object in the transverse $(9-p)$-dimensional space.

We denote $x^\mu, \mu = 0, \ldots, p$, the $(p+1)$-dimensions along the brane worldvolume, and $x^m, m = p+1, \ldots, 9$, the $(9-p)$ transverse dimensions.

These states describe objects with p spatial dimensions plus time, charged under the RR $(p+1)$-forms. The supergravity solution for N coincident Dp-branes is, for $p \leq 6$

$$ds^2 = Z(r)^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + Z(r)^{1/2}dx^m dx^m \tag{40}$$

$$e^{2\phi} = Z(r)^{(3-p)/2} \tag{41}$$

$$Z(r) = 1 + \left(\frac{\rho}{r}\right)^{7-p}, \quad \text{where} \quad \rho^{7-p} = g_s N(\alpha')^{(7-p)/2}(4\pi)^{(5-p)/2}\Gamma\left(\frac{7-p}{2}\right) \tag{42}$$

$$H_{8-p} = \frac{N}{r^{8-p}}d(\text{vol})_{\mathbb{S}^{8-p}} \tag{43}$$

where $r = \sum_m |x^m|^2$ is the radial coordinate in the transverse space $\mathbb{R}^{9-p}$.

3.1 Flux compactification

Upon compactifying a string theory, we may be left with a moduli space of parameters related to the compact geometry. The moduli are signaled by a vanishing potential and thus can take an arbitrary value. The existence of the moduli imply massless[1] scalar fields within the theory (excitations of perpendicular directions in the moduli space) which are inconsistent with observation. Thus, consistent compactifications contain no moduli. This can be accomplished by choosing physical systems where non-trivial potentials arise and constrain the would-be moduli to values of our choosing. This is known as *modulus stabilization*. The goal is to generate potentials which fix the values of all moduli to give an admissible string background or *string vacuum*. This led to the development of flux compactifications in the early 2000s. The general idea is to couple gravity to a flux which contributes to the potential that otherwise has unsatisfactory properties (collapsing to zero, expanding to infinity, etc.). In a complete string theory compactification the potential contains contributions from orientifolds, D-branes, and fluxes. In Type IIB theories, for example, the NS-NS field $B_{\mu\nu}$ and the RR field $A_{\mu\nu}$ have three-index field strengths that can provide these stabilizing fluxes as integrals over three-dimensional manifolds (exactly how the magnetic fluxes are integrals of the two-index $F_{\mu\nu}$ over two dimensional manifolds). A typical compactification on a six-dimensional Calabi-Yau space can have hundreds of independent, non-contractible, three-dimensional submanifolds, each able to support a flux characterized by an

[1]The moduli are massless due to the vanishing of the potential

integer. Thus, these fluxes can generate a potential dependent on hundreds of integers. A particular choice of integers is an example of a *flux compactification*. Work out an explicit (but simple) example.

4 AdS/CFT

Holography or AdS/CFT correspondence refers to an equivalence between a theory of gravity on anti-de Sitter (AdS) spacetime and a conformal field theory (CFT) on the boundary of AdS. Consider a field theory inside a volume V^d residing in $d+1$-dimensional spacetime. To define the field theory we perform a uniform discretization over the whole volume, creating a *lattice*. The only relevant parameter for our discussion is the size of individual lattice sites, we'll call this parameter b. The field is now defined at each lattice site by averaging the value of the field over the volume of the lattice sites. For example, say our field theory is defined inside a three-dimensional cube with side lengths B. If we discretize the volume of the cube into smaller cubes, the total number of lattice sites is simply $(B/b)^3$ (the total volume divided by the volume of a single lattice site). The dynamics of our lattice are determined from a chosen Hamiltonian, we can consider a general Hamiltonian of the form

$$H = \sum_{x,i} J_i(x,b) O^i(x) \tag{44}$$

where x denotes the location of lattice sites and i indicates a sum over the i different operators define on our lattice. $J_i(x,b)$ are the couplings (sources) for the i operators which in general may depend on the size of the lattice sites b. By varying the lattice spacing we can determine the flow of the coupling through scale transformations i.e. the so-called β-function:

$$\beta_i(J_i(x,b),x) \equiv b\frac{\partial J_i}{\partial b} \tag{45}$$

This is not too dissimilar to the effect of increasing the resolution of an image. Take, for example, a two dimensional photo with some pixel size which we allow to vary. For large pixels (lattice spacing) the photo is blurry, with no discernible structure. By increasing the resolution (decreasing the lattice spacing) objects may begin to appear and increasing further will allow the photo to be seen in all its glory. We could create operators which are pixel-dependent and tell us the color 'state' of the ith pixel. The coupling or source could be the associated 'R'-value corresponding to the RGB coloring scale. As the resolution is increased or decreased these RGB couplings will change appreciably between the lowest (IR) and highest (UV) resolutions.

In QFT, if the coupling is weak, the β-function can be determined via perturbation theory. However, at strong coupling, AdS/CFT proposes to consider the scale 'b' as another dimension. In this way, the sources $J_i(x,b)$ are considered as fields $\phi_i(x,b)$ in a space with one extra dimension whose dynamics are controlled by some action. Specifically in the AdS/CFT duality the dynamics of the ϕ_i's are determined by some theory of gravity i.e. some metric. Therefore we can consider the holographic duality as a *geometrization* of the quantum dynamics encoded by the renormalization group. The microscopic couplings of the field theory can then be identified with the values of the bulk fields at the boundary of the extra-dimensional space.

The only problem which remains is to match the degrees of freedom contained by the QFT and those at the boundary of the AdS space.

4.1 Anti-de Sitter space

Finding the geometry associated with a general QFT is a difficult problem in general. The problem is much easier if the theory has non-trivial fixed points i.e. zeroes of the β-function $\neq 0$. Once the theory evolves to the fixed point, the coupling no longer evolves and the theory becomes scale invariant. Let's consider a QFT in d-dimensions the most general metric in $d+1$-dimensions with Poincare invariance in d-dimensions is given by

$$ds^2 = \Omega^2(z)(dt^2 - d\vec{x}^2 - dz^2) \tag{46}$$

Where $\Omega(z)$ is an arbitrary function of the extra-dimension which will be fixed by the requirement of conformal symmetry. Under a conformal transformation we have

$$(t,\vec{x}) \to \lambda(t,\vec{x}), \quad z \to \lambda z \tag{47}$$

where λ is an arbitrary real number. The metric becomes

$$\Omega^2(z)(dt^2 - d\vec{x}^2 - dz^2) \to \Omega^2(\lambda z)\lambda^2(dt^2 - d\vec{x}^2 - dz^2) \tag{48}$$

To remain invariant we require

$$\Omega(z) = \frac{L}{z} \tag{49}$$

where L is a constant which we refer to as the AdS radius. The final conformally invariant metric is thus given by

$$ds^2 = \frac{L^2}{z^2}(dt^2 - d\vec{x}^2 - dz^2) \tag{50}$$

The metric, of course, has a singularity at $z = 0$ and thus we will need a regularization in order to obtain sensible results near the singularity. The AdS metric is a solution to the equations of motions for a theory of gravity with actions of the type:

$$S = \frac{1}{16\pi G_N}\int dx^{d+1}\sqrt{(-1)^{d-1}g}\left(-2\Lambda + R + c_2R^2 + c_3^3 + \cdots\right) \tag{51}$$

where G_N is the Newton constant, c_i are constants, $g = \det(g_{MN})$, R is the scalar curvature ($R \equiv R^{MN}g_{MN}$), and Λ is a cosmological constant. If $c_2 = c_3 = \cdots = 0$, we are left with our beloved Einstein-Hilbert action of general relativity. The equations of motion are given by the Einstein equations: (See Valeria for derivation of Einstein equations from the above action, I should do this once at the very least)

$$R_{MN} - \frac{1}{2}g_{MN}R = -\Lambda g_{MN} \tag{52}$$

Taking the trace

$$-\frac{(1-d)}{2}R = -(d+1)\Lambda \tag{53}$$

$$R = 2\frac{d+1}{d-1}\Lambda \tag{54}$$

Inserting back into our Einstein equations

$$R_{MN} = \left(\frac{2\Lambda}{d-1}\right)g_{MN} \tag{55}$$

Given the definition of our conformal metric in Eq.(50) we can also compute the Ricci curvature tensor directly from its definition i.e.

$$R_{\mu\nu} = R_{\mu\rho\nu}{}^{\rho} = \frac{\partial}{\partial x^\rho}\Gamma^\rho{}_{\mu\nu} - \frac{\partial}{\partial x^\mu}\Gamma^\rho_{\rho\nu} + \Gamma^\alpha{}_{\mu\nu}\Gamma^\rho{}_{\alpha\rho} - \Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\alpha\mu} \tag{56}$$

Where $R_{\mu\rho\nu}{}^{\rho}$ is the Riemann tensor and Γ is the Christoffel symbol defined by

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left[\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\rho}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right] \tag{57}$$

Plugging in our conformal metric gives (do this explicitly)

$$R_{MN} = -\frac{d}{L^2}g_{MN} \tag{58}$$

comparing the two calculation we see that

$$\frac{2\Lambda}{d-1} = -\frac{d}{L^2} \Rightarrow \Lambda = -\frac{d(d-1)}{2L^2} \tag{59}$$

which is always negative. We now see how the scalar curvature is related to the AdS_{d+1} radius

$$R = -\frac{d(d+1)}{L^2} \tag{60}$$

We see that the AdS radius is related to the amount of curvature our AdS space exhibits.

4.2 Matching degrees of freedom

We now know have identified a geometry which could act as our extra-dimensional boundary which our field theory may live on. The only sacrifice we have made thus far is the fact that we must deal with conformal field theories — so be it. We are now tasked with comparing the degrees of freedom on the boundary of our theory of gravity in $d+1$ dimensions with the degrees of freedom of our CFT. If the duality is to hold, these degrees of freedom should match exactly.

Consider our CFT to be defined in a box of volume V with some characteristic side length R, serving as our IR regulator, and some minimal lattice spacing ϵ, serving as our UV regulator. We define the number of degrees of freedom at each lattice site as c, referred to as the *central charge*. The total number of degrees of freedom in d spacetime dimensions is thus given as

$$N_{\mathrm{QFT}_d} = \left(\frac{R}{\epsilon}\right)^{d-1} c \tag{61}$$

The central charge c is one of the main defining features of a CFT. For an $SU(N)$ CFT, the gauge fields are described by $N \times N$ matrices transforming in the adjoint representation of the gauge group. Thus, we can count the degrees of freedom by considering the number of independent real parameters contained in these matrices. First, consider an $n \times n$ unitary matrix M with the following properties

1.
$$M_{ii} = M_{ii}^* \tag{62}$$

2.
$$M_{ij} = M_{ji}^* \qquad \text{for } i \neq j \tag{63}$$

Without any constraints a general $n \times n$ complex matrix has $2n^2$ independent real parameters. The first constraint reduces this number by the number of diagonal elements i.e. n. The second constraint only allows for two real parameters in the upper (or lower) diagonal or just one real parameter for each off diagonal element i.e. $n^2 - n$. Thus, the total number of free independent parameters is equal to

$$2n^2 - n - n^2 + n = n^2 \tag{64}$$

Now consider an $N \times N$ special unitary matrix M' containing one additional property

1.
$$\det(M) = 1 \rightarrow \mathrm{Tr}(M) = 0 \tag{65}$$

This places a constraint on one of the diagonal elements of M' leaving us with a total of $N^2 - 1$ degrees of freedom. Thus, for $SU(N)$ CFT's with large N the central charge scales as $c \sim N^2$.

For our theory of gravity defined in $d+1$ dimensions the degrees of freedom are characterized by the entropy. The entropy in a given volume for some theory of gravity is bounded above by the entropy of a black hole which can fit into the volume. Berkenstein and Hawking's holographic principle tells us that the entropy of a black hole is proportional to it's surface area

$$S_{\mathrm{BH}} = \frac{1}{4G_N} A_{\mathrm{BH}} \tag{66}$$

For our purposes, this tells us that the number of degrees of freedom in our AdS_{d+1} theory are thus proportional to the area A_∂ of the $z = 0$ boundary where our CFT lives

$$N_{\mathrm{AdS}_{d+1}} = \frac{A_\partial}{4G_N} \tag{67}$$

The 'surface area' of our . Because the conformal metric is singular at $z = 0$ we evaluate at the UV cutoff boundary i.e. $z = \epsilon$

$$A_\partial = \int_{\mathbb{R}_{d-1}} d^{d-1}x \sqrt{(-1)^{d-1}g}\,\delta(z-\epsilon) = \left(\frac{L}{\epsilon}\right)^2 \int d^{d-1}x \tag{68}$$

Note:

$$\sqrt{(-1)^{d-1}g} = \sqrt{(-1)^{d-1}\det g_{MN}} \tag{69}$$

$$g_{MN} = \begin{pmatrix} (L/z)^2 & 0 & 0 & \cdots \\ 0 & -(L/z)^2 & 0 & \\ 0 & 0 & -(L/z)^2 & \\ \vdots & & & \ddots \end{pmatrix}, \qquad (d-1) \times (d-1) \tag{70}$$

Thus the only contributing term comes from the multiplication of the diagonal

$$\det g_{MN} = \prod_{i=0}^{M} g_{ii} = (-1)^{d-1} \left(\frac{L}{z}\right)^{2(d-1)} \tag{71}$$

Thus,

$$A_\partial = \left(\frac{L}{\epsilon}\right)^{d-1} \int d^{d-1}x \tag{72}$$

The final integral is an integral over the remaining spacetime which is infinite. As we did for the QFT we regularize the integral by placing the theory in a box of volume V and side lengths R.

$$A_\partial = \left(\frac{LR}{\epsilon}\right)^{d-1} \tag{73}$$

To get a more direct comparison between the degrees of freedom we introduce the Planck length l_P and the Planck mass M_P for a gravity theory in $d+1$-dimensions

$$G_N = l_P^{d-1} = \frac{1}{M_P^{d-1}} \tag{74}$$

and thus

$$N_{\text{AdS}_{d+1}} = \frac{1}{4} \left(\frac{R}{\epsilon}\right)^{d-1} \left(\frac{L}{l_P}\right)^{d-1} \tag{75}$$

For the theories to be dual the degrees of freedom must match on either side,

$$\left(\frac{R}{\epsilon}\right)^{d-1} c = \frac{1}{4} \left(\frac{R}{\epsilon}\right)^{d-1} \left(\frac{L}{l_P}\right)^{d-1} \tag{76}$$

i.e.

$$c = \frac{1}{4} \left(\frac{L}{l_P}\right)^{d-1} \tag{77}$$

The classical picture of our theory of gravity defined by the action in Eq.(51) holds when the coefficient in from L^{d-1}/G_N is large, due to the increasing validity of the saddle point approximation. Our theory in the AdS_{d+1} space of radius L is thus dominated by the classical field configurations of metrics when

$$\left(\frac{L}{l_p}\right)^{d-1} \gg 1 \Rightarrow L \gg l_p \tag{78}$$

From Eq.(60) this indicates that curvature $R \propto 1/L^2$ must be small. This also indicates from Eq.(77) that a QFT has a classical gravity dual when c is large (large number of degrees of freedom per volume or large number of species). Directly:

$$4c \gg 1 \tag{79}$$

For an $SU(N)$ CFT this requires

$$4(N^2 - 1) \gg 1 \tag{80}$$

In other words, the duality holds when we are in the large N limit of our gauge theory.

5 Maldacena's lecture

Should be called quantum field theory/quantum gravity duality. Equality between a spacetime and a consider quantum gravity on the interior of the space and a gauge theory or quantum field theory on the boundary of that space. String theory is one way of quantizing gravity.

$$\text{String theory} \leftrightarrow U(N) \text{ gauge theory} \tag{81}$$

This is because the string coupling constant g is proportional to N^{-2}. Due to the the N–colors i.e. N–gluons which are able to transmit a force the true coupling of a given gauge theory is $\sim g_{\text{YM}}N \equiv \lambda$ i.e. the t'Hooft coupling. Duality is due to two parameter spaces in which the dynamics are calculable. The warp factor is simply a gravitational potential which can in general depend on the coordinates of spacetime. The energy of a particle within this well is given by

$$E = m\omega(z) \tag{82}$$

Classically, we expect the particle to live at the minimum of this potential. Quantum mechanically we expect the probability density to be localized at the minima as well with higher order excitation stemming from the higher modes of the wave–function.

Naively, this means that there are an infinite number of particles in this potential well i.e. an infinite number of Kaluza-Klein modes which may be excited. In the gravity theory, it turns out, it is possible to do a truncation on the perturbations of the fields and leave us with a finite number of fields. This comes from the holographic principle i.e. the entropy of a black hole may be described by its surface area or more practically, if you were to try and excite very high energy modes you would reach a point in which you produces a black hole.

$$N_{\text{fields}} \sim \frac{R^3}{G_N^{(5)}} \tag{83}$$

A massive point particle in the bulk at z_0 corresponds to an extended object on the boundary with a size proportional to z_0.

Why can the warp factor be thought of as a gravitational potential? Probably the simplest way to see this is to consider a point particle living in a spacetime with the following spacetime metric:

$$ds^2 = \omega(z)(dx_{1,3}^2 + dz^2) \tag{84}$$

$$S = -m \int ds \tag{85}$$

we have

$$ds = \sqrt{\omega(z)(-dt^2 + d\vec{x}^2 + dz^2)} = dt\sqrt{\omega(z)\left[\left(\frac{d\vec{x}}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]} \tag{86}$$

A simple exercise.

Another nice coordinate system for AdS

$$ds^2 = R^2\left[-d\tau^2 \cosh\rho^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2\right] \tag{87}$$

There is an important caveat with respect to a gravity dual of QCD. One of the basic assumptions in the gravity dual is that we work in a limit which suppresses (make very heavy) states with spin > 2. However, in QCD we see resonances which have spins greater than 2 and these states are not that much heavier than states with spin less than two. This means that any gravity approximation will not be arbitrarily perfect in this limit.

6 DBI Action

The world-volume action a Dp-brane is given by, in the string frame, as:

$$S_{Dp} = -\tau_p \int_{\mathcal{M}_{p+1}} d^{p+1}\xi \; e^{\Phi} \sqrt{-\det\left(\hat{G}_{ab} + \hat{B}_{ab} + 2\pi l_s^2 F_{ab}\right)} + \mu_p \int_{\mathcal{M}_{p+1}} \sum_q \hat{C}_q \wedge e^{\hat{B}+2\pi l_s^2 F} \tag{88}$$

where $\mathcal{M}_{p+1}$ denotes the full Dp-brane world-volume, ξ^a, $a = 0, \ldots, p$ parameterize the world-volume directions, τ_p is the brane tension, μ_p is the R-R charge of the D-brane, $\hat{G}_{ab}$, $\hat{B}_{ab}$, and $\hat{C}_{ab}$ are the 'pull-backs' (induced) of the space-time metric, two-form gauge potential of the NS-NS sector, and p-form potentials of the R-R sector onto the world volume of the brane i.e.

$$\hat{G}_{ab} = \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b} G_{\mu\nu} \tag{89}$$

and likewise for B and C. Equivalently in the Einstein frame we have

$$S_{Dp} = -\tau_p \int_{\mathcal{M}_{p+1}} d^{p+1}\xi \; e^{(p-3)\Phi/4} \sqrt{-\det\left(\hat{G}_{ab} + \hat{B}_{ab} + 2\pi l_s^2 F_{ab}\right)} + \mu_p \int_{\mathcal{M}_{p+1}} \sum_q \hat{C}_q \wedge e^{\hat{B}+2\pi l_s^2 F} \tag{90}$$

7 Coupling

The gauge/gravity correspondence stems from the duality between open and closed strings. The low energy dynamics of massless open strings on a Dp-brane gives rise to a $(p+1)$-dimensional SUSY gauge theory while a Dp-brane in the closed string perspective is described as a classical solution of the low-energy supergravity equations of motion charged under the R-R field C_{p+1} (boundary of the conformal field theory of closed strings). This is the basis of the gauge/gravity correspondence which we will now clarify. First, we find a background solution to a theory of supergravity. This includes finding solutions to the metric and any other p-forms which the theory may be charged under. Once the solution is obtained we can then plug this solution back into the original action, expand with a special truncation and obtain information about the dual gauge theory. In particular, we will be interested in finding the properties of the dual gauge coupling constant as well as the non-perturbative θ-angle. The truncation can be viewed as taking the leading order terms in the $\ell_s \to 0$ limit. First, we split the coordinates into two groups: those parallel and perpendicular to the world volume.

Consider N D-branes in flat Minkowski space.

7.1 $D(p+1)$-branes wrapped on two-cycles: An example

Here we derive some relevant formulae which will allow us to directly write down gauge theoretic quantities in terms of the dual supergravity parameters. First we recall the DBI action for a $D(p+2)$-brane

$$S_{Dp} = -\tau_{p+2} \int d^{p+3}\xi \; e^{-\Phi} \sqrt{-\det\left(\hat{G}_{ab} + \hat{B}_{ab} + 2\pi l_s^2 F_{ab}\right)} \tag{91}$$

Next, we split the coordinates into two groups: those which parameterize the compact two-cycle which the branes are wrapped and those which don't.

1. We define ξ^α for $\alpha = 0, \ldots, p$ which describe the 'flat'-coordinates. We will assume that this part of the metric is diagonal and that $B_{\alpha\beta} = 0$
2. ξ^A for $A = p+1, p+2$ parameterize the two-cycle. We assume $F_{AB} = 0$ and that there may be a non-trivial metric G_{AB} and B-field.

The determinant can then be factorized as

$$\det\left(\hat{G}_{ab} + \hat{B}_{ab} + 2\pi l_s^2 F_{ab}\right) = \det\left(G_{\alpha\beta} + 2\pi l_s^2 F_{\alpha\beta}\right) \det\left(G_{AB} + B_{AB}\right) \tag{92}$$

We are interested in expanding determinants of the form

$$\det(X + AB) \tag{93}$$

Using Sylvester's determinant theorem we have

$$\det(X + AB) = \det(X)\det(\mathbb{1} + BX^{-1}A) \tag{94}$$

Now we consider $B = \epsilon$ where ϵ is a small parameter $\in \mathbb{R}$ and defining $C \equiv X^{-1}A$. We are now interested in expanding a determinant of the form $\det(\mathbb{1} + \epsilon C)$. We have the following identity

$$\det(\exp[\epsilon C]) = \exp(\epsilon \mathrm{Tr}[C]) \tag{95}$$

Expanding the exponential on both sides

$$\det\left(\mathbb{1} + \epsilon C + \frac{\epsilon^2}{2} C^2 + \cdots\right) = 1 + \epsilon \mathrm{Tr} C + \frac{\epsilon^2}{2} (\mathrm{Tr} C)^2 + \cdots \tag{96}$$

Focusing on the left hand side

$$\det\left(\mathbb{1} + \epsilon C + \frac{\epsilon^2}{2}C^2 + \cdots\right) = \det\left(\mathbb{1} + \epsilon(C + \frac{\epsilon}{2}C^2 + \cdots)\right) \tag{97}$$

Using the fact

$$\det(\mathbb{1} + \epsilon C) = 1 + \epsilon \mathrm{Tr}(C) + \mathcal{O}(\epsilon^2) \tag{98}$$

we have

$$= 1 + \epsilon \mathrm{Tr}\left(C + \frac{\epsilon}{2}C^2 + \cdots\right) + \epsilon^2 f\left(C + \frac{1}{2}C^2 + \cdots\right) + \mathcal{O}(\epsilon^3) \tag{99}$$

$$= 1 + \epsilon \mathrm{Tr}\,(C) + \epsilon^2\left\{\frac{1}{2}\mathrm{Tr}(C^2) + f\left(C + \frac{1}{2}C^2 + \cdots\right)\right\} + \mathcal{O}(\epsilon^3) \tag{100}$$

Comparing powers of ϵ we find

$$f\left(C + \frac{1}{2}C^2 + \cdots\right) = \frac{(\mathrm{Tr}(C))^2 - \mathrm{Tr}(C^2)}{2} \tag{101}$$

Back to business, if we consider $2\pi l_s^2$ as our expansion parameter (i.e. consider the limit where $l_s \to 0$)

$$\det(G_{\alpha\beta}+2\pi l_s^2 F_{\alpha\beta}) = \det(G_{\alpha\beta})\left(1 + 2\pi l_s^2 \mathrm{Tr}(G^{-1}F) + \frac{(2\pi l_s^2)^2}{2}\big((\mathrm{Tr}(G^{-1}F))^2 - \mathrm{Tr}((G^{-1}F)^2)\big) + \cdots\right) \tag{102}$$

We note $G^{\alpha\beta}F_{\alpha\beta} = 0$ due to the sum over a symmetric and antisymmetric objects. The term

$$\mathrm{Tr}(G^{-1}FG^{-1}F) = \mathrm{Tr}(FF) = F^{\alpha\beta}F_{\alpha\beta} = G^{\alpha\lambda}G^{\beta\sigma}F_{\alpha\beta}F_{\lambda\sigma} \tag{103}$$

Keeping only the quadratic terms in the expansion we have

$$\approx -\tau_{p+2}\int d^{p+3}\xi e^{-\Phi}\sqrt{-\det G_{\alpha\beta}}\sqrt{1 - \frac{(2\pi l_s^2)^2}{2}\mathrm{Tr}((G^{-1}F)^2))}\sqrt{\det(G_{AB} + B_{AB})} \tag{104}$$

$$\approx -\tau_{p+2}\frac{(2\pi l_s^2)^2}{4}\int d^{p+3}\xi e^{-\Phi}\sqrt{-\det G_{\alpha\beta}}G^{\alpha\lambda}G^{\beta\sigma}F_{\alpha\beta}F_{\lambda\sigma}\sqrt{\det(G_{AB} + B_{AB})} \tag{105}$$

We note that the brane tension can be related to the string coupling and length via

$$\tau_p = \frac{1}{(2\pi)^p g_s l_s^{p+1}} \tag{106}$$

and thus define

$$g_{Dp}^2 \equiv \frac{2}{(2\pi l_s^2)\tau_p} = 2(2\pi)^{p-2} g_s l_s^{p-3} \tag{107}$$

and thus

$$g_{D(p+2)}^2 = 2(2\pi)^p g_s l_s^{p-1} \tag{108}$$

After promoting F to a non-abelian field strength, which comes with an additional normalization factor of 1/2, we have:

$$= -\frac{1}{g_{Dp}^2 (2\pi l_s)^2} \int d^{p+3}\xi e^{-\Phi} \sqrt{-\det G_{\alpha\beta}} \frac{1}{4} (F^{\alpha\beta})^a (F_{\alpha\beta})^a \sqrt{\det(G_{AB} + B_{AB})} \tag{109}$$

$$= \left[\frac{1}{g_{Dp}^2 (2\pi l_s)^2} \int d^2\xi e^{-\Phi} \sqrt{\det(G_{AB} + B_{AB})} \right] \left[\frac{1}{4} \int \mathrm{Tr}(\star F \wedge F) \right] \tag{110}$$

As it is written above, we see immediately that the gauge coupling constant is given by

$$\frac{1}{g_{\mathrm{YM}}^2} = \frac{1}{g_{Dp}^2 (2\pi l_s)^2} \int d^2\xi e^{-\Phi} \sqrt{-\det(G_{AB} + B_{AB})} \tag{111}$$

i.e. directly related to the volume of the two-cycle which the brane is wrapped.

8 Wilson Loops

See Maldacena 9803002 for the original prescription and then see Nunez, Piai, and Rago 0909.0748 for calculational details. Also see Maldacena's TASI lectures.

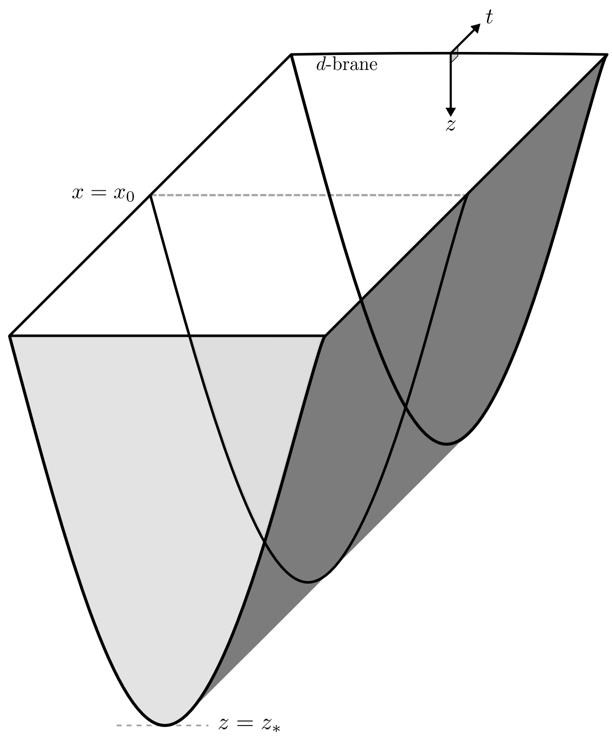

Figure 1: Hanging-string world volume

9 Quark-antiquark potential

We consider a QED-charged particle moving along a closed curve $\mathcal{C}$ with an action given by:

$$S_{\mathcal{C}} = \oint_{\mathcal{C}} A_\mu dx^\mu \tag{112}$$

This is equivalent to adding to the path integral the term

$$e^{iS_{\mathcal{C}}} \equiv W(\mathcal{C}) \tag{113}$$

The non-abelian analogue is given by

$$W(\mathcal{C}) = \mathrm{Tr} P \exp\left[i \oint_{\mathcal{C}} A_\mu dx^\mu \right] \tag{114}$$

where $A_\mu \equiv A_\mu^a T^a$, Tr is the trace over the group indices and P is the path ordering operator. We note that $\langle W(\mathcal{C}) \rangle$ can be interpreted as the amplitude for the creation of a $q\bar{q}$ pair and annihilation after traveling some distance $\propto |\mathcal{C}|$. For example, consider a rectangular loop with side lengths a and b. In the limit where one side length becomes large. The key here is that in euclidean signature the time evolution operator becomes e^{tH}, such that for a contour where $t \to \infty$ the only surviving energy state is that of the vacuum. See

10 Wilson Loops

In a gravity dual, a quark is represented by an open strings ending on a D-brane. Thus, the path $\mathcal{C}$ is represented by the boundary $\partial\Sigma$ of an open string world-sheet Σ. We have

$$\langle W(\mathcal{C})\rangle = Z_{\text{string}}(\partial\Sigma = \mathcal{C}) \tag{115}$$

For infinitely massive quarks (non-dynamical), the string length $l_s \to \infty$ pushing the D-brane to the boundary of AdS at $z = 0$. In the t'Hooft limit ($N \to \infty$) i.e. when the string action is dominated by classical field configurations, we have

$$Z_{\text{string}}(\partial\Sigma = \mathcal{C}) = e^{-S(\mathcal{C})} \tag{116}$$

Input more details, this is not exactly pedagogical

Now we are interested in computing the holographic analogue of the $q\bar{q}$ potential. We consider a rectangular contour for a sting which extends along a single coordinate x and whose endpoints reside on the boundary of AdS ($z = 0$). The induced metric in euclidean signature on the strings worldsheet can be written as

$$ds^2 = \frac{L^2}{z^2}(dt^2 + dx^2 + dz^2) = \frac{L^2}{z^2}\left[dt^2 + (1 + z'^2)dx^2\right] \tag{117}$$

where

$$z' \equiv \frac{dz}{dx} \tag{118}$$

The Nambu-Goto action is thus given by

$$S_{\text{NG}} = \frac{1}{2\pi\alpha'}\int dtdx\sqrt{g} = \frac{L^2T}{2\pi\alpha'}\int dx\frac{\sqrt{1+z'^2}}{z^2} \tag{119}$$

we now have a typical minimization problem where we can employ the tools developed for classical Lagrangian mechanics. We need to set about finding the exact shape of the catenary produced by the hanging string. We have the Euler-Lagrange equation:

$$\frac{d}{dx}\left(\frac{\partial L}{\partial z'}\right) = \frac{\partial L}{\partial z} \tag{120}$$

where

$$L = \frac{L^2 T}{2\pi\alpha'} \frac{\sqrt{1 + z'^2}}{z^2} \tag{121}$$

we obtain

$$z'' + \frac{2(1 + z'^2)}{z} = 0 \tag{122}$$

multiplying by z' and rearranging

$$\frac{z'' z'}{1 + z'^2} = -2\frac{z'}{z} \tag{123}$$

integrating both sides over dx we have

$$\ln(\sqrt{1 + z'^2}) = \ln\left(\frac{1}{z^2}\right) + C \tag{124}$$

$$1 + z'^2 = \frac{B}{z^4} \tag{125}$$

where $B \equiv e^{2C}$. Solving for z'

$$z' = \pm\sqrt{\frac{B}{z^4} - 1} \tag{126}$$

If we place our string ends at a distance d apart and place our origin at one of these ends then z must take a maximal value at $d/2$ (assuming the string ends reside on the same d-brane at $z_1 = z_2 = z_0$ maybe we can generalize to non-equal heights $z_1 \neq z_2$) and thus $z'(x = d/2) = 0$. Defining the maximal value of z as $z_* \equiv z(x = d/2)$

$$B = z_*^4 \tag{127}$$

thus,

$$z' = \pm\sqrt{\frac{z_*^4}{z^4} - 1} \tag{128}$$

Now we can integrate and solve for $x(z)$

$$x(z) = \int_z^{z_*} dz \left(\sqrt{\frac{z_*^4}{z^4} - 1}\right)^{-1} \tag{129}$$

we make a substitution

$$u = \frac{z}{z_*}, \quad du = \frac{1}{z_*} dz \tag{130}$$

$$x(z) = z_* \int_{z/z_*}^{1} du \frac{u^2}{\sqrt{1-u^4}} \tag{131}$$

given the boundary conditions we can solve for z_* as a function of the distance d between the end points on the d-brane

$$d = z_* \int_0^1 du \frac{u^2}{\sqrt{1-u^4}} \tag{132}$$

which yields

$$d = z_* \frac{\sqrt{\pi}\Gamma\left(\frac{7}{4}\right)}{3\Gamma\left(\frac{5}{4}\right)} \tag{133}$$

$$z_* = 1.67d \tag{134}$$

Now that we have parameterized the catenary, we need to compute the "on-shell" action for the string from which we can extract the $q\bar{q}$ potential. After reinserting our results into the NG action we have

$$S_{\text{NG}} = \frac{L^2 T z_*^2}{2\pi\alpha'} \int dx \frac{1}{z^4} \tag{135}$$

Transform coordinates

$$dx = \left(\sqrt{\frac{z_*^4}{z^4} - 1}\right)^{-1} dz \tag{136}$$

$$S_{\text{NG}} = 2 \times \frac{L^2 T z_*^2}{2\pi\alpha'} \int_0^{z_*} \frac{dz}{z^2\sqrt{z_*^4 - z^4}} \tag{137}$$

Performing another u-substitution

$$S_{\text{NG}} = \frac{L^2 T}{\pi\alpha'} \frac{1}{z_*} \int_0^1 \frac{du}{u^2\sqrt{1-u^4}} \tag{138}$$

This integral is divergent on the interval from 0 to 1, however, we can rewrite the integral with the lower bound starting away from 0

$$S_{\text{NG}} = \frac{L^2 T}{\pi\alpha'} \frac{1}{z_*} \int_{\epsilon/z_*}^1 \frac{du}{u^2\sqrt{1-u^4}} \tag{139}$$

and for small ϵ it can be shown that

$$S_{\text{NG}} = \frac{L^2 T}{\pi\alpha'} \frac{1}{z_*} \left(-\frac{\pi^{3/2}\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)^2} + \frac{z_*}{\epsilon}\right) \tag{140}$$

The on shell or classical action is just given by ET and thus we have the energy of the quark-antiquark pair as

$$E = -\frac{4\pi^2 L^2}{\Gamma\left(\frac{1}{4}\right)^4 \alpha'}\frac{1}{d} + \frac{L^2}{\pi\alpha'}\frac{1}{\epsilon} \tag{141}$$

The divergent piece corresponds to the quark and antiquark masses (which we have considered to be infinite). To check this definitively we can compute the action of a single string string which goes straight from the boundary at $z = \epsilon$ to $z = \infty$ at a fixed value of x. The euclidean metric is given by

$$\frac{L^2}{z^2}(dt^2 + dz^2) \tag{142}$$

The action is given then by

$$S_{|} = \frac{TL^2}{2\pi\alpha'}\int_{\epsilon}^{\infty}\frac{dz}{z^2} = \frac{TL^2}{\pi\alpha'}\frac{1}{\epsilon} \tag{143}$$

for two parallel strings we have

$$S_{||} = 2\times\frac{TL^2}{\pi\alpha'}\frac{1}{\epsilon} \tag{144}$$

matching exactly the divergent term in our expression. The true potential energy between the two quarks is thus given by

$$E_{q\bar{q}} = E - E_{||} = -\frac{4\pi^2 L^2}{\Gamma\left(\frac{1}{4}\right)^4 \alpha'}\frac{1}{d} \tag{145}$$

$$= -0.11\frac{L^2}{\alpha'}\frac{1}{d} \tag{146}$$

in terms of gauge theory parameters where from Eq.(**??**) $L^2 = \sqrt{N^2 g_{\rm YM}}\alpha'$ or in terms of the t'Hooft coupling $L^2 = \sqrt{\lambda}\alpha'$ we have

$$E_{q\bar{q}} = -\frac{4\pi^2\sqrt{\lambda}}{\Gamma\left(\frac{1}{4}\right)^4}\frac{1}{d} = -0.23\sqrt{\lambda}\frac{1}{d} \tag{147}$$

Comparatively, the perturbative $q\bar{q}$ potential (for $\lambda \ll 1$) is given by

$$E_{q\bar{q}}^{\lambda\ll 1} = -\pi\lambda\frac{1}{d} \tag{148}$$

For a more general treatment we assume the same set up with the string residing on the boundary of AdS and extending in only a single coordinate x. We can however use the more general ansatz for the metric

$$ds^2 = f(z)\left(dt^2 + (1+z'^2)dx^2\right) \tag{149}$$

The action is thus given by

$$S_{\text{NG}} = \frac{T}{2\pi\alpha'}\int dx f(z)\sqrt{1+z'^2} \tag{150}$$

This gives us a Lagrangian of the form

$$L = \frac{T}{2\pi\alpha'} f(z)\sqrt{1+z'^2} \tag{151}$$

The Euler-Lagrange equation gives us a relation between z' as a function of f

$$1 + z'^2 = Bf(z)^2 \tag{152}$$

where B is an arbitrary constant to be determined from boundary conditions. This as far as we can get without an explicit expression for $f(z)$ but to continue from here we would solve for z', perform a separation of variables, and integrate to solve for x as a function of z.

10.1 An even more general treatment

See Nunez, Piai, Rago: Wilson Loops in String Duals of Walking Technicolor
We will consider the generic background given by

$$ds^2 = -g_{tt}dt^2 + g_{xx}d\vec{x}^2 + g_{\rho\rho}d\rho^2 + g_{ij}d\theta^i d\theta^j \tag{153}$$

and assume the functions $g_{tt}, g_{xx}, g_{\rho\rho}$, and g_{ij} depend only on the radial coordinate ρ. In principle the function g_{ij} for the internal coordinates could also depend on other coordinates, however, in this analysis we choose a configuration for a probe string that is not excited on the θ^i directions.

The induced metric on the string world-sheet is given by

$$ds^2 = -g_{tt}dt^2 + g_{xx}dx^2 + g_{\rho\rho}d\rho^2 \tag{154}$$

The action is given by

$$S_{\text{NG}} = \frac{T}{2\pi\alpha'} = \int dx\sqrt{F^2 + G^2\rho'^2} \tag{155}$$

where we have defined

$$F^2 \equiv g_{tt}g_{xx}, \quad G^2 \equiv g_{tt}g_{\rho\rho} \tag{156}$$

The resulting equations of motion from the extracted Lagrangian result in the following differential equation

$$\rho'' + \left(\frac{G'}{G} - \frac{2F'}{F}\right)\rho' - \frac{FF'}{G^2} = 0 \tag{157}$$

Likewise, because the Lagrangian doesn't depend on x explicitly it must be a conserved charge. We thus have

$$\rho'\frac{\partial L}{\partial \rho'} - L = C \tag{158}$$

where C is an arbitrary constant. We have the relation

$$-\frac{F^2}{\sqrt{F^2 + G^2\rho'^2}} = C \tag{159}$$

$$\rho'^2 = \frac{F^2(F^2 - C^2)}{C^2G^2} \tag{160}$$

This gives a direct relation between x and the functions $F(\rho)$ and $G(\rho)$. The constant C can be obtained by considering the turning point as the maximal value of ρ, as done above. Defining ρ_* as the maximal value of ρ i.e. $\rho(x = L/2) = \rho_*$ we have

$$C = F_* \equiv F(\rho_*) \tag{161}$$

Now we define

$$V(\rho) = \frac{F\sqrt{F^2 - F_*^2}}{F_*G} \tag{162}$$

and obtain

$$\left|x(\rho) - \frac{L}{2}\right| = \int_{\rho_0}^{\rho} \frac{d\rho}{V(\rho)} \tag{163}$$

where L is he distance between the $q\bar{q}$ pair. The $q\bar{q}$ potential is obtained by plugging our results back into the action, extracting the energy piece, and subtracting the divergent contribution stemming from infinite mass of the quarks i.e. the string configuration corresponding to two rods extending from the d-brane to infinity.

10.2 Effective Theory and Wilson Loops

Generically, effective theory allows us to describe the low energy or IR dynamics of a field theory approximately independently of the UV physics. For example, effective field theory methods provided as a very powerful tool in excluding theories beyond the Standard Model. In the context of holography, effective field theory methods have been applied in numerous areas See Solana, Gutiez, Hoyos.

For large values of the $q\bar{q}$ separation L, intuition tells us that the shape of the string near the AdS boundary at $\rho = 0$ should consist of essentially perpendicular strings to the boundary. These straight segments may be thought of as UV contributions to the IR physics which are essentially fixed and non-changing up to a value of the radial coordinate ρ_* where the geometry begins to deform the hanging string configuration. We may then, be able to write the large distance potential between the quarks independently of the UV geometry and in terms of the 'cutoff' radial coordinate ρ_*. The UV geometry will then be encoded into coefficients multiplying the $q\bar{q}$ potential. By differentiating with respect to our cutoff scale ρ_* we can write holographic renormalization equations describing how the contributions from the UV geometry flow to the IR, in other words, these are the holographic β–functions.

See Nunez, Piai, Rago: Wilson Loops in String Duals of Walking Technicolor
We will consider a generic 10–dimensional background metric given by

$$ds^2 = -g_{tt}dt^2 + g_{xx}d\vec{x}^2 + g_{rr}dr^2 + g_{ij}d\theta^i d\theta^j \tag{164}$$

and assume the functions g_{tt}, g_{xx}, g_{rr}, and g_{ij} depend only on the radial coordinate r. In principle the function g_{ij} for the internal coordinates could also depend on other coordinates, however, in this analysis we won't consider probe string excitations along the compact θ^i directions. Here r corresponds to the radial coordinate with the boundary (UV) approached as $r \to \infty$ and the end of the space (IR) reached at some critical value r_0. Orienting our $q\bar{q}$ pair along a single spatial dimension in field theory space for simplicity gives the following induced metric on the string world-sheet

$$ds^2 \approx -g_{tt}dt^2 + g_{xx}dx^2 + g_{rr}dr^2. \tag{165}$$

In the gravity dual (and in the t'Hooft limit), obtaining the potential energy between the $q\bar{q}$ pair as a function of separation distance L amounts to solving for the classical catenary curve associated with a string whose ends are stuck to the boundary separated at a distance L within the space described by Eq.(165). The Nambu-Goto action for the string is given by

$$S_{\rm NG} = \frac{T}{2\pi\alpha'}\int dx\sqrt{F^2 + G^2 r'^2} \tag{166}$$

where we have defined $r' \equiv dr/dx$, $T = \int dt$ and

$$F^2 \equiv g_{tt}g_{xx}, \quad G^2 \equiv g_{tt}g_{rr}. \tag{167}$$

Using either the integrated equations of motion or the conservation of the Hamiltonian results in the following differential equation [TM: Adapt discussion for multi-warp setups]

$$r'^2 = \frac{F^2(F^2 - C^2)}{C^2G^2} \tag{168}$$

where C is an arbitrary constant. The constant C can be obtained by considering the boundary conditions in which the two string ends are separated by some distance L along the field theory coordinate. The string extends into the bulk until hitting its turning point at a minimal value of the radial coordinate r_*. Because we assume a symmetric space we have $r(x = L/2) = r_*$. This gives

$$C = F(r_*) \equiv F_* \tag{169}$$

After defining

$$V(r) = \frac{F\sqrt{F^2 - F_*^2}}{F_*G} \tag{170}$$

the geodesic equation is given by

$$\frac{dr}{dx} = \pm V. \tag{171}$$

where the two solutions describe either side of r_* ($-V$ for $x < L/2$, $+V$ for $x > L/2$). The separation length L between the string endpoints may now be written as

$$L = \int dx = 2\int_{r_*}^{\infty} \frac{dr}{V} \tag{172}$$

The energy of the string configuration is given by the length of the string as seen from the bulk

$$E' = \int \mathcal{L}dx = 2\int_{r_*}^{\infty} \frac{F^2}{F_*}\frac{dr}{V} = \frac{1}{2}F_*L + \int_{r_*}^{\infty}\left(\frac{G}{F}\sqrt{F^2 - F_*^2} + \frac{1}{2}\frac{F^2}{F_*}\frac{1}{V}\right)dr \tag{173}$$

$$= F_*L + 2\int_{r_*}^{\infty}\frac{G}{F}\sqrt{F^2 - F_*^2}dr. \tag{174}$$

In general the energy of this class of configurations are divergent and require regularizing. The divergent piece corresponds the infinite masses of the 'quarks' and can be regularized by adding a counter-term $\delta E_{||}$ corresponding to the energy of two strings extending from the boundary at $r = \infty$ to the horizon at $r = r_0$ i.e.

$$\delta E_{||} = 2 \int_{r_0}^{\infty} G dr. \tag{175}$$

The regularized energy between the $q\bar{q}$ pair is thus given by

$$E = F_* L + 2 \int_{r_*}^{\infty} \frac{G}{F} \sqrt{F^2 - F_*^2} dr - 2 \int_{r_0}^{\infty} G dr \tag{176}$$

$$\equiv F_* L + 2K_*. \tag{177}$$

As the separation distance between the string ends increases, the string falls deeper and deeper into the space. For sufficiently large separation we can introduce a cutoff parameter at $r = \rho$ such that $x(\rho) \equiv \delta x \ll L$. In other words, we introduce a cutoff scale such that the contribution above the cutoff are those from approximately straight string segments up to corrections of $\mathcal{O}(\delta x)$. This allows us to factorize the action into three parts: 1. the UV contribution to the string $S^>$ stemming from the two approximately straight string segments above the cutoff ρ and 2. the IR contribution $S^<$ stemming from the non-trivial string configuration below the cutoff.

$$S_{\mathrm{NG}} = S^> + S^< \tag{178}$$

We write $S^<$ by expanding in x' to quadratic order

$$S^> = \frac{T}{2\pi\alpha'} \int_{\rho}^{\infty} \left[G + \frac{F^2}{2G} x'^2 + \mathcal{O}(x'^4) \right] dr \tag{179}$$

utilizing the conservation of the generalized momentum with respect to x' we find

$$x' = \frac{G}{F} \sqrt{2 \left(1 + \frac{\tilde{C}}{G} \right)} \equiv \tilde{V} \tag{180}$$

where $\tilde{C}$ is an arbitrary constant. This leaves us with the on-shell action

$$S^> = \frac{T}{2\pi\alpha'} \int_{\rho}^{\infty} \left(2G + \tilde{C} \right) dr. \tag{181}$$

The first term can be interpreted as a partial contribution to the total quark masses or rods extending from the horizon while the second term proportional to x'^2 encodes corrections due to the small curvature of the string proportional to δx with respect to the cutoff i.e. $\tilde{C}$ encodes $x(\rho)$. The length on the boundary field theory at the cutoff ρ in the radial direction is given by[2]

$$L^{>} = 2\int dx = 2\int_{\rho}^{\infty} \tilde{V} dr \equiv 2\delta x. \tag{182}$$

The IR string action is thus given by

$$S_{\rm IR} = S^{<} + \frac{T}{2\pi\alpha'}\int_{\rho}^{\infty}\left(2G + \tilde{C}\right) dr. \tag{183}$$

with the length between the string ends expressed as

$$L = 2\delta x + 2\int_{r_*}^{\rho}\frac{dr}{V} \tag{184}$$

and energy given by

$$E = F_* L' + \int_{r_*}^{\rho}\frac{G}{F}\sqrt{F^2 - F_*^2}dr - 2\int_{r_0}^{\rho} G dr + \tilde{C}L^{>} - 2\int_{\rho}^{\infty} G(1-2\tilde{V})dr \tag{185}$$

$$\begin{aligned} E &= F_* L' + \int_{r_*}^{\rho}\frac{G}{F}\sqrt{F^2 - F_*^2}dr - 2\int_{r_0}^{\rho} G dr - 2\int_{\rho}^{\infty} G(1-2\tilde{V}-\tilde{C})dr && (186)\\ &= E_{\rm IR} - K && (187)\end{aligned}$$

where L' is to be understood as the length between the string ends at the cutoff ρ and likewise $F_* = F(x = L'/2) = F(x = L/2)$. We see that the IR potential is equivalent to the new string system hanging from $r = \rho$ and modified by a constant term proportional the quark mass contribution plus some corrections proportional to the slope of the string x' between the boundary and cutoff. The β-functions can also be extracted.

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[2]Note that there is a key importance in which variable we choose to play with. If we write our Lagrangian in terms of r' then, because F and G depend on r we must use the conservation of the Hamiltonian i.e. $\sum_i \dot{q}_i p_i - \mathcal{L} =$ constant where $p \equiv \partial\mathcal{L}/\partial r'$. If we write our Lagraingian in terms of x' we can directly use the fact that the generalized momentum $\partial\mathcal{L}/\partial x' =$ constant.

10.3 Determining bulk geometry from Wilson Loops

[TM: See Hashimoto: Building bulk from Wilson Loops] This is nothing other than inverting the relations obtained above. Could pose as a very interesting method to determine if the Wilson loop data from a given gauge theory, say on the lattice, has a gravity dual.

10.4 Hadronization

Discuss the Lund string model and how the gauge-string duality may be used to compute dE/dx or the equivalent fragmentation function.

11 Entanglement Entropy

11.1 Determining bulk geometry from entanglement entropy

[TM: See Hashimoto, Watanabe: Bulk reconstruction of metrics inside black holes by complexity]

12 Relating Wilson Loops and Entanglement Entropy

[TM: See Nunez, Sonnenchein, et al.]

13 Peskin-Takeuchi S parameter

See Anguelova 1006.3570 for derivation as well Sonnenschein the original papers from Peskin and Takeuchi.

As shown above, the Peskin-Takeuchi S-parameters may be written as

$$S = -4\pi \frac{d}{dq}(\Pi_V - \Pi_A)\bigg|_{q^2=0} \tag{188}$$

This can be rewritten as a sum over vector and axial-vector resonances as

$$S = 4\pi \sum_n \left(\frac{g_{V_n}^2}{m_{V_n}^4} - \frac{g_{A_n}^2}{m_{A_n}^4} \right) \tag{189}$$

To compute this quantity from a holographic perspective amounts to computing the couplings and masses shown in Eq.(189).

14 θ-parameter

15 Cosmology

It is interesting to consider how our four-dimensional world may result from the description of a traveling three-brane embedded in a ten dimensional string theory. The resulting cosmology can be obtained by considering the geodesic of the three-brane through the bulk of a higher dimensional space.

For now, we'll assume a probe three-brane and will show that the motion in ambient space induces cosmological expansion or contraction simulating "matter" or a cosmological constant (inflation). This is known as "mirage cosmology" i.e. where the cosmological expansion is due to the constraints induced by the bulk background geometry.

Here we consider a probe $D3$-brane moving along a geodesic in a general static, spherically-symmetric background. Because the brane moves along a geodesic, the induced world-volume becomes a function of time such that the "observers" on the $D3$ brane experience a changing (expanding or contracting) universe.

It is interesting to ask — what type of generic background geometry mimics the expansion we see today in the standard cosmological model? Is there a string background which already exists that gives the desired properties?

The metric may be parameterized as

$$ds^2 = -g_{tt}(r)dt^2 + g_{xx}(r)d\vec{x}^2 + g_{rr}(r)dr^2 + g_{\theta\theta}(r)d\Omega \tag{190}$$

where we will parameterize the compact coordinates via a spherical metric i.e. $d\Omega \equiv h_{ij}\theta^i\theta^j$ and the metric components $g_{tt}, g_{xx}, g_{rr}, g_\theta$ are all functions of the radial coordinate r. The probe $D3$-brane follows its geodesic governed by the DBI action in Eq.(34)

$$S_{D3} = -T_3 \left\{ \int_{\mathcal{M}_4} d^4\xi \ \left[e^{-\Phi}\sqrt{-\det\left(\hat{G}_{ab} + \hat{B}_{ab} + 2\pi l_s^2 F_{ab}\right)} - \sum_q \hat{C}_q \wedge e^{\hat{B}+2\pi l_s^2 F} \right] \right\} \tag{191}$$

$$= -T_3 \left\{ \int_{\mathcal{M}_4} d^4\xi \ \left[e^{-\Phi} \sqrt{-\det\left(\hat{G}_{ab} + \hat{B}_{ab} + 2\pi l_s^2 F_{ab}\right)} - \hat{C}_4 \right] \right\} \tag{192}$$

where again

$$\hat{G}_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b} \tag{193}$$

we are free to choose the convenient *static* gauge $x^a = \xi^a$ for $a = 0, 1, 2, 3$. Assuming, to start, $\hat{B}_{ab} = F_{ab} = 0$ we have

$$\begin{aligned} \hat{G}_{00} &= -g_{tt} + g_{rr}\dot{r}^2 + g_{\theta\theta} h_{ij} \dot{\theta}^i \dot{\theta}^j \\ \hat{G}_{ij} &= g_{xx} \delta_{ij} \text{ for } i, j = 1, 2, 3 \end{aligned} \tag{194}$$

where we have used the static condition $\dot{\vec{x}} = 0$ and defined

$$\dot{r} \equiv \frac{\partial r}{\partial t}, \quad \dot{\theta} = \frac{\partial \theta}{\partial t} \tag{195}$$

The Lagrangian density is thus given by

$$\mathcal{L} = -e^{\Phi} \sqrt{-g_{xx}^3 \left(-g_{tt} + g_{rr}\dot{r}^2 + g_{\theta\theta} h_{ij} \dot{\theta}^i \dot{\theta}^j\right)} + C \tag{196}$$

For generality we define

$$A(r) = -e^{-2\Phi} g_{xx}^3 g_{tt}, \quad B(r) = -e^{-2\Phi} g_{xx}^3 g_{rr}, \quad D(r) = -e^{-2\Phi} g_{xx}^3 g_{\theta\theta} \tag{197}$$

giving

$$\mathcal{L} = -\sqrt{-A + B\dot{r}^2 + D g_{ij} \dot{\theta}^i \dot{\theta}^j} + C \tag{198}$$

Both r and θ are independent of the world-volume coordinates and thus must be constants along the motion. We have

$$\sum_i \dot{q}_i \frac{\partial \mathcal{L}}{\dot{q}_i} - \mathcal{L} = K \tag{199}$$

we find

$$\frac{-A - \frac{1}{2} D g_{ij} \dot{\theta}^i \dot{\theta}^j (1 - \delta_{ij})}{\sqrt{-A + B\dot{r}^2 + D g_{ij} \dot{\theta}^i \dot{\theta}^j}} + C = K \tag{200}$$

if the compact coordinates are compactified on an n-sphere, we have $g_{ij} = 0$ for $i \neq j$

$$\frac{-A}{\sqrt{-A + B\dot{r}^2 + D\dot{\theta}^2}} + C = K \tag{201}$$

where it is understood $\dot{\theta}^2 \equiv g_{ii}\dot{\theta}^i\dot{\theta}^i$. We also have the generalized momentum

$$p_i = -\frac{D g_{ij}\dot{\theta}^j}{\sqrt{-A + B\dot{r}^2 + D g_{ij}\dot{\theta}^i\dot{\theta}^j}} \equiv -L \tag{202}$$

where L is the total angular momentum

$$p_i p^i = g^{ij} p_i p_j = L^2 = \frac{D^2 g^{ij} g_{ik} g_{jl}\dot{\theta}^k\dot{\theta}^l}{-A + B\dot{r}^2 + D g_{ij}\dot{\theta}^i\dot{\theta}^j} = \frac{D^2\dot{\theta}^2}{-A + B\dot{r}^2 + D\dot{\theta}^2} \tag{203}$$

substituting into Eq.(201) and solving for $\dot{\theta}^2$ we find

$$\dot{\theta}^2 = \frac{A^2 L^2}{D^2 (K - C)^2} \tag{204}$$

Plugging this result back in we can then determine the evolution of r

$$\dot{r}^2 = \frac{A}{B}\left[1 + \frac{A}{(K - C)^2}\left(1 - \frac{L^2}{D}\right)\right] \tag{205}$$

On the probe 3-brane we have the following induced four-dimensional metric

$$d\hat{s}^2 = (-g_{tt} + g_{rr}\dot{r} + g_{ij}\dot{\theta}^i\dot{\theta}^j)dt^2 + g_{xx}d\vec{x}^2 \tag{206}$$

Plugging in Eq.(197) we obtain

$$d\hat{s}^2 = \frac{-e^{-2\Phi} g_{xx}^3 g_{tt}^2}{(K - C)^2} dt^2 + g_{xx}d\vec{x}^2 \tag{207}$$

Defining the cosmological time (proper time of the 3-brane universe)

$$d\eta = \frac{e^{-\Phi} g_{xx}^{3/2} |g_{tt}|}{|K - C|} dt \tag{208}$$

we have

$$d\hat{s}^2 = -d\eta^2 + g_{xx}d\vec{x}^2 \tag{209}$$

This is the standard form for a flat, expanding universe! By defining the scale factor as $a^2 \equiv g_{xx}$ we can derive the analogous Friedmann equations:

15.1 p-branes traveling in p'-brane backgrounds

15.2 Branes with constant B fields

16 Holography at Finite Temperature

16.1 Transport coefficients

16.2 Dyson-Schwinger Equations

In quantum field theories all physical content is stored in correlation functions.

$$\langle \phi_i(x_1)\phi_j(x_2)\cdots\phi_k(x_n)\rangle = \int \mathcal{D}\phi_i(x_1)\phi_j(x_2)\cdots\phi_k(x_n)e^{-S[\phi_i]} \tag{210}$$

The 'usual' calculation of n-point correlation functions involves a perturbative expansion powers of an infinitesimal parameter. Of course this procedure fails when we have to deal with couplings which are no longer $\ll 1$. In these cases we need to turn to non-perturbative methods to calculate correlation functions. One such method resulting in a non-linear first-order functional differential equation was developed independently by Dyson and Schwinger. The core entity in these calculations is the effective action $\Gamma[\Phi_i]$. The effective action is defined as

$$\Gamma[\Phi] \equiv \sup_J \left(\int [-W[J] + \Phi_i J_i] \right) \tag{211}$$

Where the J_i's denote sources for the fields (operators) Φ_i and $W[J]$ is the generating functional of connected correlators. The generating functional is related to the bare action $S[\phi]$ via the path integral

$$Z[J] \equiv e^{W[J]} = \int \mathcal{D}[\phi]e^{-S[\phi]+\phi_j J_j} \tag{212}$$

All n-point correlation functions can be generated by taking functional derivatives of the generating functional with respect to the sources.

$$\langle \phi_i(x_1)\phi_j(x_2)\cdots\phi_k(x_n)\rangle = \frac{1}{Z[0]}\left[\frac{\delta^n Z[J]}{\delta J_i(x_1)\delta J_j(x_2)\cdots\delta J_k(x_n)}\right]_{J=0} \tag{213}$$

Where ϕ_i denotes the quantum fields. At J_{sup} the variation of the effective action must be zero and thus,

$$\frac{\delta\Gamma_{\text{sup}}}{\delta J_j(x)} = 0 = \frac{\delta}{\delta J_j(x)}\left(\int -W[J] + \Phi_i J_i\right) \tag{214}$$

$$\frac{\delta W[J]}{\delta J_j(x)} = \Phi_i \delta_{ij} \tag{215}$$

But,

$$\frac{\delta W[J]}{\delta J_j(x)} = \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J_j(x)} = \langle \phi(x) \rangle_J \tag{216}$$

Thus the average field Φ is given by

$$\Phi_i \equiv \langle \phi_i \rangle_J = \frac{\delta W}{\delta J_i} = Z[J]^{-1} \int \mathcal{D}[\phi] \phi_i e^{-S[\phi] + \phi_i J_i} \tag{217}$$

Noting that

$$\frac{\delta \Gamma_{\text{sup}}[\Phi]}{\delta \Phi_i(x)} = -\int \frac{\delta W}{\delta J_j(y)} \frac{\delta J_j(y)}{\delta \Phi_i(x)} dy + \int \frac{\delta \Phi_k(y)}{\delta \Phi_i(x)} J_k dy + \int \frac{\delta J_l(y)}{\delta \Phi_i(x)} \Phi_l dy \tag{218}$$

$$= -\Phi_j \delta_{ji} + J_k \delta_{ki} + \Phi_l \delta_{li} = J_i(x) \tag{219}$$

We can now write down a functional differential equation for the effective action considering Eq.(212)

$$e^{-\Gamma[\Phi]} = \int \mathcal{D}[\phi] \exp\left[-S[\phi + \Phi] + \frac{\delta \Gamma[\Phi]}{\delta \Phi_i} \phi_i\right] \tag{220}$$

An exact solution for $\Gamma[\Phi]$ is difficult to obtain but we can perform a vertex expansion of $\Gamma[\Phi]$

$$\Gamma[\Phi] = \sum_{n=0}^{\infty} \frac{1}{\mathcal{N}^{i_1 \cdots i_n}} \sum_{i_1 \cdots i_n} \int d^D x_1 \cdots d^D x_n \Gamma^{i_1 \cdots i_n} \Phi_{i_1}(x_1) \cdots \Phi_{i_n}(x_n) \tag{221}$$

Where $\mathcal{N}$ is the corresponding symmetry factor and $\Gamma^{i_1 \cdots i_n}$ correspond to the one-particle irreducible proper vertices. Inserting this back into Eq.(220) and comparing the coefficients of the field monomials results in an infinite tower of coupled integro-differential equations for $\Gamma^{i_n \cdots i_n}$, these are the Dyson-Schwinger equations.

16.3 Wilson Loops at Finite Temperature

16.4 Viscosity

See Edelstein, Portugues Sect. 5.9 as well as Ramallo Intro to AdS/CFT. Following papers are also relevant 0309213, 0405231, 0011179, and 0209163

References