SU(6) NON-RELATIVISTIC QUARK MODEL

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The following notes are split into four parts. In the first part I give a brief introduction to SU(6). In the second part I calculate the baryon octet wave functions in order to calculate magnetic moments. In the third section I use the Gursey-Radicati mass formula to calculate the mass splitting between the Σ and Λ. In the fourth section I use SU(6) wave functions to calculate the axial vector form factor from neutron beta decay. Finally, I end with concluding remarks.

SU(6): An Introduction

SU(3) symmetry ignores all internal degrees of freedom such as quark spin, angular momentum, etc. An obvious extension to SU(3) is the inclusion of one of these internal degrees of freedom. As you may guess, this can be done by allowing each quark to take on one of two values of spin ($\pm 1/2$) and takes us into a six-dimensional tensor product space with a flavor index i which runs from 1 to 3 and a spin index j which runs from 1 to 2. This leads us directly into the special unitary group in 6 dimensions (SU(6)) defined as

$$SU(6) \equiv \{u| \text{ complex } 6{\times}6 \text{ matrices, satisfying } u^\dagger u = \mathbb{1}_{6\times 6}, \det(u) = \mathbb{1}\} \tag{1}$$

SU(6) is a semi-simple compact Lie group of rank 5 with $6^2 - 1 = 35$ unique generators which span the entire space (Appendix: for a rainy day). The two fundamental representations $[\mathbf{6}]$ and $[\bar{\mathbf{6}}]$

$$[\mathbf{6}] = \{q_{ij}\} = \{q_{11}, q_{12}, q_{21}, q_{22}, q_{31}, q_{32}\} = \{u^\uparrow, u^\downarrow, d^\uparrow, d^\downarrow, s^\uparrow, s^\downarrow\} \tag{2}$$

$$[\bar{\mathbf{6}}] = \{\bar{q}_{ij}\} = \{\bar{q}_{11}, \bar{q}_{12}, \bar{q}_{21}, \bar{q}_{22}, \bar{q}_{31}, \bar{q}_{32}\} = \{\bar{u}^\uparrow, \bar{u}^\downarrow, \bar{d}^\uparrow, \bar{d}^\downarrow, \bar{s}^\uparrow, \bar{s}^\downarrow\} \tag{3}$$

We can form Kronecker products of the fundamental representations and reduce them furth to reveal the SU(6) multiplets. We can make this more explicit by expressing the products with the quantum numbers of the subgroup SU(3)×SU(2)⊂SU(6). We have

$$^{[\mathbf{6}]}SU(6) \rightarrow^{[\{\mathbf{3}\},\frac{1}{2}]} SU(3) \times SU(2) \tag{4}$$

$$^{[\bar{\mathbf{6}}]}SU(6) \rightarrow^{[\{\bar{\mathbf{3}}\},\frac{1}{2}]} SU(3) \times SU(2) \tag{5}$$

We obtain the meson multiplets from the direct product

$$[\mathbf{6}] \otimes [\bar{\mathbf{6}}] = [\mathbf{1}] \oplus [\mathbf{35}] \tag{6}$$

Denoting the spin explicitly we find

$$[\{\mathbf{3}\}, \frac{1}{2}] \otimes [\{\bar{\mathbf{3}}\}, \frac{1}{2}] = [\{\mathbf{1}\}, 0] \oplus [\{\mathbf{1}\}, 1] \oplus [\{\mathbf{8}\}, 1] \oplus [\{\mathbf{8}\}, 0] \tag{7}$$

Which gives us the expected results of the spin-1 vector mesons and spin-0 psuedoscalar mesons. We obtain the baryon multiplets by the direct product of 3 fundamental representations

$$[\mathbf{6}] \otimes [\mathbf{6}] \otimes [\mathbf{6}] = [\mathbf{20}] \oplus [\mathbf{56}] \oplus [\mathbf{70}] \oplus [\mathbf{70}] \tag{8}$$

We can further decompose the $[\mathbf{56}]$

$$[\mathbf{56}] = [\{\mathbf{8}\}, \frac{1}{2}] \oplus [\{\mathbf{10}\}, \frac{3}{2}] \tag{9}$$

Which we recognize as the baryon octect (N, Σ, Ξ, $\cdots$) and decuplet of the baryon resonances $(\Delta, \Sigma^*, \cdots)$

You may notice that we find more states than physically observed in the direct product of the three quark sextets. It is thought that the remaining states in the $[\mathbf{20}], [\mathbf{70}]_1, [\mathbf{70}]_2$ have very high masses and cannot be observed with current collider energies.

PART I: Magnetic Moments of the Baryon Octet

To simplify calculations we will say that there is no orbital angular momentum contribution to the wave function (l=0). Thus, we will be dealing with particles in the ground state whose angular momentum is given completely by their spin. This makes the calculation of the total magnetic moment much more palatable. All we need to do is add up the spin contributions from each constituent quark. We make the educated guess that the total magnetic moment operator will be the sum of the magnetic moments of each individual quark

$$\hat{\mu} = \sum_{i}^{3} \mu_q(i)\hat{\sigma}(i) \tag{10}$$

Where i denotes the ith quark and $\hat{\sigma}(i)$ is the spin operator acting on the ith quark. The magnetic moment of hadrons can be measured experimentally along a particular direction (Stern-Gerlach Experiment). As normal convention we choose to calculate the magnetic moment along the z-axis. Thus in calculating the expected value for the z-component of the magnetic moment for any hadron state $|h\rangle$, we have

$$\langle h| \hat{\mu} |h\rangle = \langle h| \sum_{i}^{3} \mu_q(i)\hat{\sigma}_z(i) |h\rangle \tag{11}$$

We see that in order to calculate the expectation value of magnetic moment we will need to first calculate the wave functions for the baryon octet.

Calculating Octet wave functions [1]

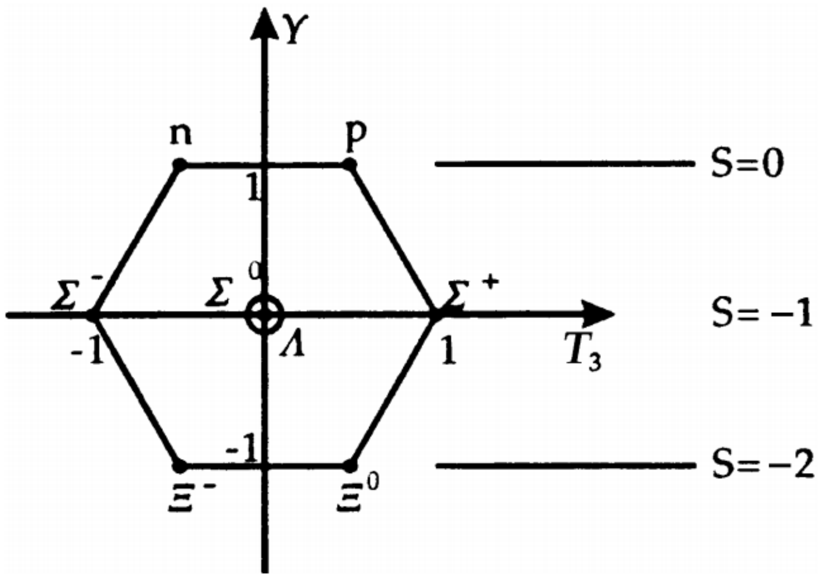

Figure 1: The baryon octet [2]

The general procedure for calculating the octet (Fig. 1) wave functions begins by constructing the highest weight state. In this case we start with the proton state. Once we have calculated the normalized wave function, we can use the shift operators (Fig. 2) defined by the groups sub-algebras to maneuver around the octet. There is a degenerate state at the center of the octet so we will have to consider this case separately when we arrive.

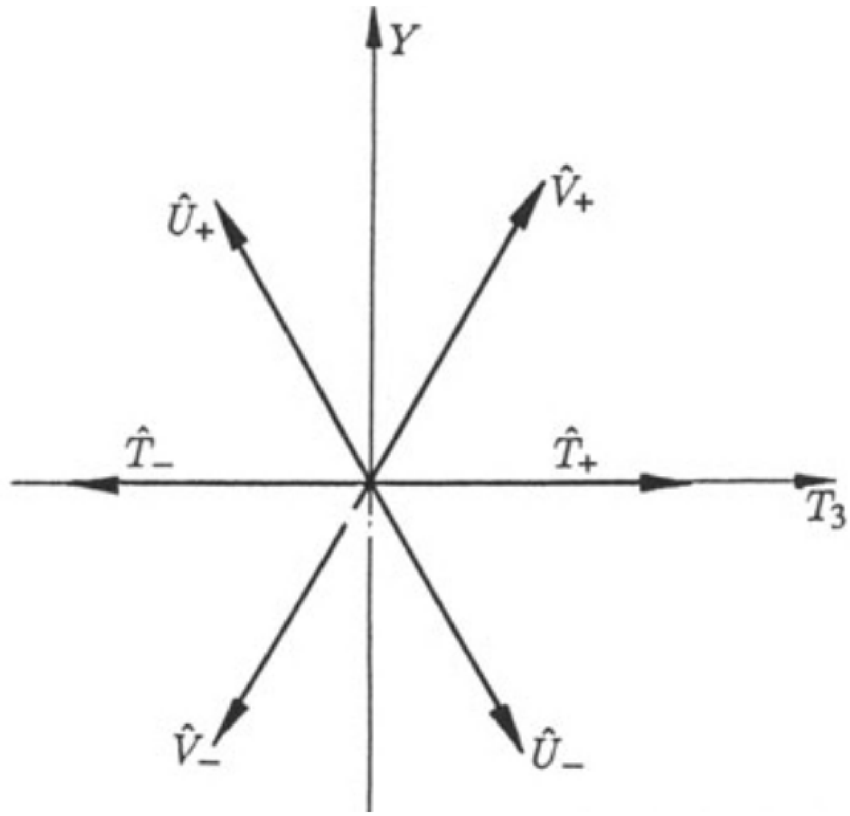

Figure 2: The shift operators which allow us to manuever around the SU(3)⊂SU(6) multiplets [1]

We define each baryon state uniquely with quantum numbers of hypercharge (Y), isospin (T), third component of isospin (T_3), total spin (J), and the spin projection (J_z). We know that the baryons are made from a linear superposition of three individual quarks. Baryon wave functions then take the general form (without spin)[1]

$$\psi_B(\mathbf{r}) = \sum_i \lambda_i \left(q_{t^1,t_3^1,y^1}(\mathbf{r}_1) \cdot q_{t^2,t_3^2,y^2}(\mathbf{r}_2) \cdot q_{t^3,t_3^3,y^3}(\mathbf{r}_3) \right)_i \tag{12}$$

Where $q_{t^j,t_3^j,y^j}(\mathbf{r}_j)$ represents a quark with quantum numbers t, t_3, y at some position $\mathbf{r}$, $(\cdots)_i$ indicates the sum over possible quark states given by the selection rules, and

[1]It is important to note that each of these quarks is located at a distinct point in spacetime ($\mathbf{r_i}$), this becomes important when we symmeterize the wave function. I eventually drop the (r) for notational brevity

λ_i are arbitrary complex numbers. To include spin, we now couple the above baryon wave function with the spin wave function

$$\psi_B(\mathbf{r}) \otimes \chi(\mathbf{r}) = \sum_i \lambda_i \left(q_{t^1,t_3^1,y^1}(\mathbf{r}_1) \cdot q_{t^2,t_3^2,y^2}(\mathbf{r}_2) \cdot q_{t^3,t_3^3,y^3}(\mathbf{r}_3) \right)_i \otimes \sum_j \lambda_j \left(\chi_{j^1,m^1}(\mathbf{r}_1) \cdot \chi_{j^2,m^2}(\mathbf{r}_2) \cdot \chi_{j^3,m^3}(\mathbf{r}_3) \right)_j$$

After expanding the product we are left with

$$\psi_B(\mathbf{r}) \otimes \chi(\mathbf{r}) = \sum_k \lambda_k \Big(q_{t^1,t_3^1,y^1}(\mathbf{r}_1) \otimes \chi_{j^1,m^1}(\mathbf{r}_1) \cdot q_{t^2,t_3^2,y^2}(\mathbf{r}_2) \otimes \chi_{j^2,m^2}(\mathbf{r}_2) \cdot q_{t^3,t_3^3,y^3}(\mathbf{r}_3) \otimes \chi_{j^3,m^3}(\mathbf{r}_3) \Big)_k \tag{13}$$

Where $\chi_{j^k,m^k}(\mathbf{r}_j)$ represent the spin eignenfunctions. We will refer to the quark states from table (1).

q_i	Y	T	T_3	Q
q_1 (u)	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$
q_2 (d)	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{3}$
q_3 (s)	$-\frac{2}{3}$	0	0	$-\frac{1}{3}$

Directly from SU(6) construction we know that the quarks each have total spin $s = \frac{1}{2}$ and thus spin projection $m_s = \pm\frac{1}{2}$. We begin by constructing the spin $+\frac{1}{2}$ proton. A proton is made from two up quarks and one down quark. To build the wave function we start by coupling two up quarks together and then coupling the down quark to the two up quark state. There are two potential uu couplings which leave us with a $J_z = +\frac{1}{2}$ spin projection. The state $(uu)^{J=1}_{J_z=1} = |u\uparrow u\uparrow\rangle$ coupled with the the $|d\downarrow\rangle$ as well as $|u\uparrow\rangle$ and $|u\downarrow\rangle$ together to form the $(uu)^{J=1}_{J_z=0}$ state and then coupled to the $|d\uparrow\rangle$ state both give spin projection $J_z = +\frac{1}{2}$. The wave function will have to be some superposition of these states. [2]. Writing it explicitly we have

$$\psi^{(uu)}_{1,1,\frac{2}{3},1,J_z} = \sum_{m,m'} \left(\frac{1}{2}, \frac{1}{2}, 1 \middle| \frac{1}{2}, \frac{1}{2}, 1 \right) \left(\frac{1}{2}, \frac{1}{2}, 1 \middle| \pm\frac{1}{2}, \pm\frac{1}{2}, J_z \right) \psi_{\frac{1}{2},\frac{1}{2},\frac{1}{3}}(1) \psi_{\frac{1}{2},\frac{1}{2},\frac{1}{3}}(2) \chi_{\frac{1}{2},m}(1) \chi_{\frac{1}{2},m'}(2)$$

[2] Because we are working in the same SU(3) multiplet the respective SU(3) isoscalar factor is 1

Where $(\frac{1}{2},\frac{1}{2},\ 1|\frac{1}{2},\frac{1}{2},\ 0) \equiv (t,t',T|t_3,t_3,T_3)$ are the respective Clebsch-Gordon coefficients (CGC) for $\mathrm{SU}_I(2)$ and $(\frac{1}{2},\frac{1}{2},1|\pm\frac{1}{2},\pm\frac{1}{2},J_z) \equiv (j,j',J|j_z,j'_z,J_z)$ are the respective CGC's for $\mathrm{SU}_s(2)$. For J_z=0 we have

$$\psi^{(uu)}_{1,1,\frac{2}{3},1,0} = \left(\frac{1}{2},\frac{1}{2},1\middle|-\frac{1}{2},\frac{1}{2},0\right)u(1)u(2)\chi_{\frac{1}{2},-\frac{1}{2}}(1)\chi_{\frac{1}{2},\frac{1}{2}}(2) + \left(\frac{1}{2},\frac{1}{2},1\middle|\frac{1}{2},-\frac{1}{2},0\right)u(1)u(2)\chi_{\frac{1}{2},\frac{1}{2}}(1)\chi_{\frac{1}{2},-\frac{1}{2}}(2) \tag{14}$$

With the respective CGC we end up with

$$\psi^{(uu)}_{1,1,\frac{2}{3},1,0} = \sqrt{\frac{1}{2}}\left(|u\uparrow\rangle|u\downarrow\rangle + |u\downarrow\rangle|u\uparrow\rangle\right) \tag{15}$$

For J_z=1

$$\psi^{(uu)}_{1,1,\frac{2}{3},1,1} = \left(\frac{1}{2},\frac{1}{2},1|\frac{1}{2},\frac{1}{2},1\right)u(1)u(2)\chi_{\frac{1}{2},1\frac{1}{2}}(1)\chi_{\frac{1}{2},\frac{1}{2}}(2) \tag{16}$$

Giving us the state

$$\psi^{(uu)}_{1,1,\frac{2}{3},1,1} = \left(|u\uparrow\rangle|u\uparrow\rangle\right) \tag{17}$$

Now we couple this with the down quark state. We are left with

$$\psi_{T=\frac{1}{2},T_3=\frac{1}{2},J=\frac{1}{2},J_z=\frac{1}{2}} = |p\uparrow\rangle = \left(1,\frac{1}{2},\frac{1}{2}\middle|0,\frac{1}{2},\frac{1}{2}\right)\sqrt{\frac{1}{2}}\left(|u\uparrow\rangle|u\downarrow\rangle|d\uparrow\rangle + |u\downarrow\rangle|u\uparrow\rangle|d\uparrow\rangle\right) + \left(1,\frac{1}{2},\frac{1}{2}\middle|1,-\frac{1}{2},\frac{1}{2}\right)\left(|u\uparrow\rangle|u\uparrow\rangle|d\downarrow\rangle\right) \tag{18}$$

$$= -\sqrt{\frac{1}{3}}\sqrt{\frac{1}{2}}\left(|u\uparrow\rangle|u\downarrow\rangle|d\uparrow\rangle + |u\downarrow\rangle|u\uparrow\rangle|d\uparrow\rangle\right) + \sqrt{\frac{2}{3}}\left(|u\uparrow\rangle|u\uparrow\rangle|d\downarrow\rangle\right) \tag{19}$$

$$= \sqrt{\frac{2}{3}}\left(|u\uparrow\rangle|u\uparrow\rangle|d\downarrow\rangle\right) - \sqrt{\frac{1}{6}}\left(|u\uparrow\rangle|u\downarrow\rangle|d\uparrow\rangle + |u\downarrow\rangle|u\uparrow\rangle|d\uparrow\rangle\right) \tag{20}$$

The normalization of the above wave function is correct however because baryons have even parity we demand that the wave function be symmetric under interchange of any two quarks. We are able to symmeterize any arbitrary wave function using the symmetrization operator $\hat{S}$ which in and of itself is made up of permutation operators $\hat{P}$. For a three body system

$$\hat{S}_{123} = \mathbb{1} + \hat{P}_{12} + \hat{P}_{13} + \hat{P}_{23} + \hat{P}_{13}\hat{P}_{12} + \hat{P}_{12}\hat{P}_{13} \tag{21}$$

Graphically these operators can be understood

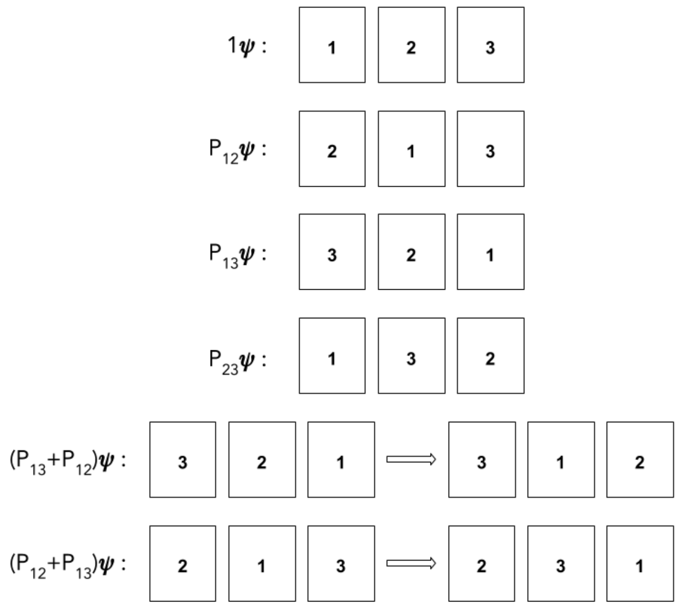

Figure 3: The permutation operator P_{ij} swaps the i and jth particle positions

Acting on $|p\uparrow\rangle$

$$\mathbb{1}\,|p\uparrow\rangle = \frac{1}{\sqrt{6}}\left(2\,|u\uparrow\rangle\,|u\uparrow\rangle\,|d\downarrow\rangle - |u\uparrow\rangle\,|u\downarrow\rangle\,|d\uparrow\rangle - |u\downarrow\rangle\,|u\uparrow\rangle\,|d\uparrow\rangle\right)$$

$$\hat{P}_{12}\,|p\uparrow\rangle = \frac{1}{\sqrt{6}}\left(2\,|u\uparrow\rangle\,|u\uparrow\rangle\,|d\downarrow\rangle - |u\downarrow\rangle\,|u\uparrow\rangle\,|d\uparrow\rangle - |u\uparrow\rangle\,|u\downarrow\rangle\,|d\uparrow\rangle\right)$$

$$\hat{P}_{13}\,|p\uparrow\rangle = \frac{1}{\sqrt{6}}\left(2\,|d\downarrow\rangle\,|u\uparrow\rangle\,|u\uparrow\rangle - |d\uparrow\rangle\,|u\downarrow\rangle\,|u\uparrow\rangle - |d\uparrow\rangle\,|u\uparrow\rangle\,|u\downarrow\rangle\right)$$

$$\hat{P}_{23}\,|p\uparrow\rangle = \frac{1}{\sqrt{6}}\left(2\,|u\uparrow\rangle\,|d\downarrow\rangle\,|u\uparrow\rangle - |u\uparrow\rangle\,|d\uparrow\rangle\,|u\downarrow\rangle - |u\downarrow\rangle\,|d\uparrow\rangle\,|u\uparrow\rangle\right)$$

$$\hat{P}_{13}\hat{P}_{12}\,|p\uparrow\rangle = \frac{1}{\sqrt{6}}\left(2\,|u\uparrow\rangle\,|d\downarrow\rangle\,|u\uparrow\rangle - |u\downarrow\rangle\,|d\uparrow\rangle\,|u\uparrow\rangle - |u\uparrow\rangle\,|d\uparrow\rangle\,|u\downarrow\rangle\right)$$

$$\hat{P}_{12}\hat{P}_{13}\,|p\uparrow\rangle = \frac{1}{\sqrt{6}}\left(2\,|d\downarrow\rangle\,|u\uparrow\rangle\,|u\uparrow\rangle - |d\uparrow\rangle\,|u\uparrow\rangle\,|u\downarrow\rangle - |d\uparrow\rangle\,|u\downarrow\rangle\,|u\uparrow\rangle\right)$$

Streamlining notation

$$\hat{S}_{123}\,|p\uparrow\rangle = N\cdot\frac{2}{\sqrt{6}}\Big(2\,|u^\uparrow u^\uparrow d^\downarrow\rangle + 2\,|d^\downarrow u^\uparrow u^\uparrow\rangle + 2\,|u^\uparrow d^\downarrow u^\uparrow\rangle - |u^\uparrow u^\downarrow d^\uparrow\rangle$$
$$- |u^\downarrow u^\uparrow d^\uparrow\rangle - |d^\uparrow u^\downarrow u^\uparrow - |d^\uparrow u^\uparrow u^\downarrow\rangle - |u^\downarrow d^\uparrow u^\uparrow\rangle - u^\uparrow d^\uparrow u^\downarrow\rangle\Big)$$

Choosing $\langle p^\uparrow|p^\uparrow\rangle = 1$ then $N = \frac{1}{\sqrt{12}}$ leaving us with the final normalized, completely symmetric proton wave function

$$|p^\uparrow\rangle = \frac{1}{\sqrt{18}}\Big(2\,|u^\uparrow u^\uparrow d^\downarrow\rangle + 2\,|d^\downarrow u^\uparrow u^\uparrow\rangle + 2\,|u^\uparrow d^\downarrow u^\uparrow\rangle - |u^\uparrow u^\downarrow d^\uparrow\rangle$$
$$- |u^\downarrow u^\uparrow d^\uparrow\rangle - |d^\uparrow u^\downarrow u^\uparrow - |d^\uparrow u^\uparrow u^\downarrow\rangle - |u^\downarrow d^\uparrow u^\uparrow\rangle - u^\uparrow d^\uparrow u^\downarrow\rangle\Big) \tag{22}$$

Now we need to construct all the other wave functions for the baryon octet. Luckily we don't need to repeat the above process, using the raising and lowering operators (Fig. 1, 2) we can maneuver around the multiplet and construct all other wave functions.

Many body operator

Before we continue on using the shift operators to manuever around the octet we need to understand how to manipulate them. Any operator which acts in the space of many-body functions (like the shift operators) is a many body operator in that

$$\hat{O}_B = \hat{O}_1 + \hat{O}_2 + \hat{O}_3$$

So, when this operator acts on a multi-body function it only acts on its respective space coordinate ($\mathbf{r}_i$). For example

$$\hat{O}_B\psi_b(\mathbf{r}) = \sum_i \lambda_i \hat{O}_B\left(q_1(\mathbf{r}_1)q_2(\mathbf{r}_2)q_3(\mathbf{r}_3)\right)_i$$

$$= \sum_i \lambda_i\left[(\hat{O}_1 q_1(\mathbf{r}_1))q_2(\mathbf{r}_2)q_3(\mathbf{r}_3) + q_1(\mathbf{r}_1)(\hat{O}_2 q_2(\mathbf{r}_2))q_3(\mathbf{r}_3) + q_1(\mathbf{r}_1)q_2(\mathbf{r}_2)(\hat{O}_3 q_3(\mathbf{r}_3))\right]_i$$

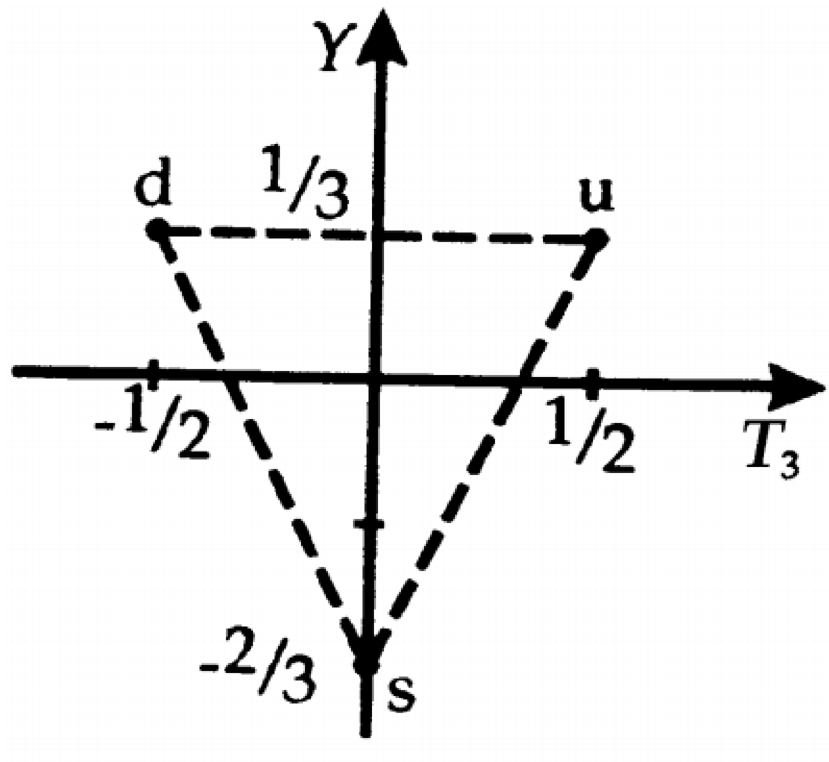

Figure 4: Smallest irreducible representation in SU(3)

Next let's construct the neutron wave function. The neutron is an isodoublet with the proton so we can us the isospin lowering operator $\hat{T}_-$(Fig. 2) to obtain its wave function.

$$|n^{\uparrow}\rangle = N \cdot \hat{T}_- |p^{\uparrow}\rangle \tag{23}$$

How does the isospin operator act on a an individual quark state? We see from the SU(3) irreducible representation (Fig. 4)

$$\hat{T}_- |u^{\uparrow}\rangle = |d^{\uparrow}\rangle \tag{24}$$

$$\hat{T}_- |d^{\uparrow}\rangle = \hat{T}_- |d^{\downarrow}\rangle = 0 \tag{25}$$

Acting on the proton state we find

$$
\begin{aligned}
|n^{\uparrow}\rangle = N \cdot \frac{1}{\sqrt{18}} \Big[& 2\,|\left(\hat{T}_{-}(1)u^{\uparrow}\right) u^{\uparrow} d^{\downarrow}\rangle + 2\,|u^{\uparrow}\left(\hat{T}_{-}(2)u^{\uparrow}\right) d^{\downarrow}\rangle + 2\,|u^{\uparrow} u^{\uparrow}\left(\hat{T}_{-}(3)d^{\downarrow}\right)\rangle \\
& + 2\,|\left(\hat{T}_{-}(1)d^{\downarrow}\right) u^{\uparrow} u^{\uparrow}\rangle + 2\,|d^{\downarrow}\left(\hat{T}_{-}(2)u^{\uparrow}\right) u^{\uparrow}\rangle + 2\,|d^{\downarrow} u^{\uparrow}\left(\hat{T}_{-}(3)u^{\uparrow}\right)\rangle \\
& + 2\,|\left(\hat{T}_{-}(1)u^{\uparrow}\right) d^{\downarrow} u^{\uparrow}\rangle + 2\,|u^{\uparrow}\left(\hat{T}_{-}(2)d^{\downarrow}\right) u^{\uparrow}\rangle + 2\,|u^{\uparrow} d^{\downarrow}\left(\hat{T}_{-}(3)u^{\uparrow}\right)\rangle \\
& - |\left(\hat{T}_{-}(1)u^{\uparrow}\right) u^{\downarrow} d^{\uparrow}\rangle - |u^{\uparrow}\left(\hat{T}_{-}(2)u^{\downarrow}\right) d^{\uparrow}\rangle - |u^{\uparrow} u^{\downarrow}\left(\hat{T}_{-}(3)d^{\uparrow}\right)\rangle \\
& - |\left(\hat{T}_{-}(1)u^{\downarrow}\right) u^{\uparrow} d^{\uparrow}\rangle - |u^{\downarrow}\left(\hat{T}_{-}(2)u^{\uparrow}\right) d^{\uparrow}\rangle - |u^{\downarrow} u^{\uparrow}\left(\hat{T}_{-}(3)d^{\uparrow}\right)\rangle \\
& - |\left(\hat{T}_{-}(1)d^{\uparrow}\right) u^{\downarrow} u^{\uparrow}\rangle - |d^{\uparrow}\left(\hat{T}_{-}(2)u^{\downarrow}\right) u^{\uparrow}\rangle - |d^{\uparrow} u^{\downarrow}\left(\hat{T}_{-}(3)u^{\uparrow}\right)\rangle \\
& - |\left(\hat{T}_{-}(1)d^{\uparrow}\right) u^{\uparrow} u^{\downarrow}\rangle - |d^{\uparrow}\left(\hat{T}_{-}(2)u^{\uparrow}\right) u^{\downarrow}\rangle - |d^{\uparrow} u^{\uparrow}\left(\hat{T}_{-}(3)u^{\downarrow}\right)\rangle \\
& - |\left(\hat{T}_{-}(1)u^{\downarrow}\right) d^{\uparrow} u^{\uparrow}\rangle - |u^{\downarrow}\left(\hat{T}_{-}(2)d^{\uparrow}\right) u^{\uparrow}\rangle - |u^{\downarrow} d^{\uparrow}\left(\hat{T}_{-}(3)u^{\uparrow}\right)\rangle \\
& - |\left(\hat{T}_{-}(1)u^{\uparrow}\right) d^{\uparrow} u^{\downarrow}\rangle - |u^{\uparrow}\left(\hat{T}_{-}(2)d^{\uparrow}\right) u^{\downarrow}\rangle - |u^{\uparrow} d^{\uparrow}\left(\hat{T}_{-}(3)u^{\downarrow}\right)\rangle \Big]
\end{aligned} \tag{26}
$$

After implementing (26) and combing like terms we are left with

$$
\begin{aligned}
|n^{\uparrow}\rangle = & \frac{N}{\sqrt{18}} \Big[|d^{\uparrow} u^{\uparrow} d^{\downarrow}\rangle + |u^{\uparrow} d^{\uparrow} d^{\downarrow}\rangle + |d^{\downarrow} d^{\uparrow} u^{\uparrow}\rangle + |d^{\downarrow} u^{\uparrow} d^{\uparrow}\rangle \\
& + |d^{\uparrow} d^{\downarrow} u^{\uparrow}\rangle + |u^{\uparrow} d^{\downarrow} d^{\uparrow}\rangle - 2\,|d^{\uparrow} u^{\downarrow} d^{\uparrow}\rangle - 2\,|u^{\downarrow} d^{\uparrow} d^{\uparrow}\rangle - 2\,|d^{\uparrow} d^{\uparrow} u^{\downarrow}\rangle \Big]
\end{aligned} \tag{27}
$$

We need to find the normalization constant for the wave function, we demand that

$$
\langle n^{\uparrow} | n^{\uparrow}\rangle = \int \psi_n^{\dagger} \psi_n d\mathbf{V} = 1 \tag{28}
$$

$$
\int \psi_n^{\dagger} \psi_n d\mathbf{V} = \prod_i^3 \int q^{*}_{t'^i, t'^i_3, y'^i} \chi^{\dagger}_{\frac{1}{2}, m'^i_s} q_{t_i, t^i_3, y^i} \chi_{\frac{1}{2}, m^i_s} d^3\mathbf{r}_i \tag{29}
$$

$$
= \prod_i^3 \delta_{t^i, t'^i} \delta_{t^i_3, t'^i_3} \delta_{y^i, y'^i} \delta_{m^i, m'^i} \tag{30}
$$

Using this result and plugging in the neutron wave function we find

$$
\langle n^{\uparrow} | n^{\uparrow}\rangle = \frac{N^2}{18}(4 \cdot 3 + 6) = 1 \tag{31}
$$

$$N = \pm 1 \tag{32}$$

Choosing N=+1 we are left with the normalized, symmetric, spin up neutron wave function

$$\begin{aligned} |n^{\uparrow}\rangle =& \frac{1}{\sqrt{18}}\bigg[|d^{\uparrow}u^{\uparrow}d^{\downarrow}\rangle + |u^{\uparrow}d^{\uparrow}d^{\downarrow}\rangle + |d^{\downarrow}d^{\uparrow}u^{\uparrow}\rangle + |d^{\downarrow}u^{\uparrow}d^{\uparrow}\rangle \\ &+ |d^{\uparrow}d^{\downarrow}u^{\uparrow}\rangle + |u^{\uparrow}d^{\downarrow}d^{\uparrow}\rangle - 2\,|d^{\uparrow}u^{\downarrow}d^{\uparrow}\rangle - 2\,|u^{\downarrow}d^{\uparrow}d^{\uparrow}\rangle - 2\,|d^{\uparrow}d^{\uparrow}u^{\downarrow}\rangle \bigg] \end{aligned} \tag{33}$$

We now proceed by constructing the other spin one-half baryon wave functions

By applying the $\hat{U}_{-}$ shift operator on the proton state we can obtain the $\Sigma^{+\uparrow}$ wave function. We need to know how the $\hat{U}_{-}$ acts on the quark states.

$$\hat{U}_{-}\,|d^{\uparrow}\rangle = |s^{\uparrow}\rangle$$

$$\hat{U}_{-}\,|d^{\downarrow}\rangle = |s^{\downarrow}\rangle$$

$$\hat{U}_{-}\,|u^{\uparrow}\rangle = \hat{U}_{-}\,|u^{\downarrow}\rangle = 0$$

It can seen that the Σ^{+} state is exactly the same as the proton state however the $|d^{(\uparrow\downarrow)}\rangle$ eigenstate is replaced with the $|s^{(\uparrow\downarrow)}\rangle$. With the condition that $\langle\Sigma^{+}|\Sigma^{+}\rangle = 1$ we obtain

$$\begin{aligned} |\Sigma^{+\uparrow}\rangle =& \frac{1}{\sqrt{18}}\bigg(2\,|u^{\uparrow}u^{\uparrow}s^{\downarrow}\rangle + 2\,|s^{\downarrow}u^{\uparrow}u^{\uparrow}\rangle + 2\,|u^{\uparrow}s^{\downarrow}u^{\uparrow}\rangle - |u^{\uparrow}u^{\downarrow}s^{\uparrow}\rangle \\ &- |u^{\downarrow}u^{\uparrow}s^{\uparrow}\rangle - |s^{\uparrow}u^{\downarrow}u^{\uparrow}\rangle - |s^{\uparrow}u^{\uparrow}u^{\downarrow}\rangle - |u^{\downarrow}s^{\uparrow}u^{\uparrow}\rangle - |u^{\uparrow}s^{\uparrow}u^{\downarrow}\rangle \bigg) \end{aligned} \tag{34}$$

Now we construct the $|\Sigma^{0\uparrow}\rangle$ wavefunction. Noting before that

$$\hat{T}_{-}\,|s\rangle = \hat{T}_{-}\,|d\rangle = 0$$

$$\hat{T}_{-}\,|u\rangle = |d\rangle$$

We find

$$\begin{aligned} \Sigma^{0\uparrow} = \hat{T}_{-}\,|\Sigma^{+\uparrow}\rangle =& \frac{N}{\sqrt{18}}\cdot\bigg(2\,|d^{\uparrow}u^{\uparrow}s^{\downarrow}\rangle + 2\,|u^{\uparrow}d^{\uparrow}s^{\downarrow}\rangle + 2\,|s^{\downarrow}d^{\uparrow}u^{\uparrow}\rangle + 2\,|s^{\downarrow}u^{\uparrow}d^{\uparrow}\rangle + 2\,|d^{\uparrow}s^{\downarrow}u^{\uparrow}\rangle \\ &+ 2\,|u^{\uparrow}s^{\downarrow}d^{\uparrow}\rangle - |d^{\uparrow}u^{\downarrow}s^{\uparrow}\rangle - |u^{\uparrow}d^{\downarrow}s^{\uparrow}\rangle - |d^{\downarrow}u^{\uparrow}s^{\uparrow}\rangle - |u^{\downarrow}d^{\uparrow}s^{\uparrow}\rangle - |s^{\uparrow}d^{\downarrow}u^{\uparrow}\rangle \\ &- |s^{\uparrow}u^{\downarrow}d^{\uparrow}\rangle - |s^{\uparrow}d^{\uparrow}u^{\downarrow}\rangle - |s^{\uparrow}u^{\uparrow}d^{\downarrow}\rangle - |d^{\downarrow}s^{\uparrow}u^{\uparrow}\rangle - |u^{\downarrow}s^{\uparrow}d^{\uparrow}\rangle - |d^{\uparrow}s^{\uparrow}u^{\downarrow}\rangle \\ &- |u^{\uparrow}s^{\uparrow}d^{\downarrow}\rangle \bigg) \end{aligned} \tag{35}$$

Demanding that $\langle \Sigma^{+\uparrow} | \Sigma^{+\uparrow} \rangle = 1$ we find that $N = \sqrt{\frac{1}{2}}$ and thus

$$\begin{aligned}\Sigma^{0\uparrow} =& \frac{1}{6} \cdot \Big(2 |d^\uparrow u^\uparrow s^\downarrow\rangle + 2 |u^\uparrow d^\uparrow s^\downarrow\rangle + 2 |s^\downarrow d^\uparrow u^\uparrow\rangle + 2 |s^\downarrow u^\uparrow d^\uparrow\rangle + 2 |d^\uparrow s^\downarrow u^\uparrow\rangle \\ & + 2 |u^\uparrow s^\downarrow d^\uparrow\rangle - |d^\uparrow u^\downarrow s^\uparrow\rangle - |u^\uparrow d^\downarrow s^\uparrow\rangle - |d^\downarrow u^\uparrow s^\uparrow\rangle - |u^\downarrow d^\uparrow s^\uparrow\rangle - |s^\uparrow d^\downarrow u^\uparrow\rangle \\ & - |s^\uparrow u^\downarrow d^\uparrow\rangle - |s^\uparrow d^\uparrow u^\downarrow\rangle - |s^\uparrow u^\uparrow d^\downarrow\rangle - |d^\downarrow s^\uparrow u^\uparrow\rangle - |u^\downarrow s^\uparrow d^\uparrow\rangle - |d^\uparrow s^\uparrow u^\downarrow\rangle \\ & - |u^\uparrow s^\uparrow d^\downarrow\rangle \Big)\end{aligned} \tag{36}$$

Again acting $\hat{T}_-$ to obtain the wave function for $|\Sigma^{-\uparrow}\rangle$

$$\begin{aligned}|\Sigma^{-\uparrow}\rangle = \hat{T}_- |\Sigma^{0\uparrow}\rangle =& \frac{N}{6} \cdot \Big(2 |d^\uparrow d^\uparrow s^\downarrow\rangle + 2 |d^\uparrow d^\uparrow s^\downarrow\rangle + 2 |s^\downarrow d^\uparrow d^\uparrow\rangle + 2 |s^\downarrow d^\uparrow d^\uparrow\rangle + 2 |d^\uparrow s^\downarrow d^\uparrow\rangle \\ & + 2 |d^\uparrow s^\downarrow d^\uparrow\rangle - |d^\uparrow d^\downarrow s^\uparrow\rangle - |d^\uparrow d^\downarrow s^\uparrow\rangle - |d^\downarrow d^\uparrow s^\uparrow\rangle - |d^\downarrow d^\uparrow s^\uparrow\rangle - |s^\uparrow d^\downarrow d^\uparrow\rangle \\ & - |s^\uparrow d^\downarrow d^\uparrow\rangle - |s^\uparrow d^\uparrow d^\downarrow\rangle - |s^\uparrow d^\uparrow d^\downarrow\rangle - |d^\downarrow s^\uparrow d^\uparrow\rangle - |d^\downarrow s^\uparrow d^\uparrow\rangle - |d^\uparrow s^\uparrow d^\downarrow\rangle \\ & - |d^\uparrow s^\uparrow d^\downarrow\rangle \Big)\end{aligned}$$

$$\begin{aligned}=& \frac{N}{6} \cdot \Big(4 |d^\uparrow d^\uparrow s^\downarrow\rangle + 4 |s^\downarrow d^\uparrow d^\uparrow\rangle + 4 |d^\uparrow s^\downarrow d^\uparrow\rangle - 2 |d^\uparrow d^\downarrow s^\uparrow\rangle - 2 |d^\downarrow d^\uparrow s^\uparrow\rangle \\ & - 2 |s^\uparrow d^\downarrow d^\uparrow\rangle - 2 |s^\uparrow d^\uparrow d^\downarrow\rangle - 2 |d^\downarrow s^\uparrow d^\uparrow\rangle - 2 |d^\uparrow s^\uparrow d^\downarrow\rangle \Big)\end{aligned} \tag{37}$$

With the normalization condition we find that $N = \frac{1}{\sqrt{2}}$ hence,

$$\begin{aligned}|\Sigma^{-\uparrow}\rangle = & \frac{1}{\sqrt{18}} \cdot \Big(2 |d^\uparrow d^\uparrow s^\downarrow\rangle + 2 |s^\downarrow d^\uparrow d^\uparrow\rangle + 2 |d^\uparrow s^\downarrow d^\uparrow\rangle - |d^\uparrow d^\downarrow s^\uparrow\rangle - |d^\downarrow d^\uparrow s^\uparrow\rangle \\ & - |s^\uparrow d^\downarrow d^\uparrow\rangle - |s^\uparrow d^\uparrow d^\downarrow\rangle - |d^\downarrow s^\uparrow d^\uparrow\rangle - |d^\uparrow s^\uparrow d^\downarrow\rangle \Big)\end{aligned} \tag{38}$$

Applying $\hat{U}_-$ to $|\Sigma^{-\uparrow}\rangle$ gives the $|\Xi^{-\uparrow}\rangle$ state. From figures (2) and (4) we see

$$\begin{aligned}\hat{U}_- |d\rangle &= |s\rangle \\ \hat{U}_- |u\rangle = \hat{U}_- |s\rangle &\equiv 0\end{aligned} \tag{39}$$

We find

$$|\Xi^{-\uparrow}\rangle = \hat{U}_- |\Sigma^{-\uparrow}\rangle = \frac{N}{\sqrt{18}} \cdot \Big[2|s^\uparrow d^\uparrow s^\downarrow\rangle + 2|d^\uparrow s^\uparrow s^\downarrow\rangle + 2|s^\downarrow s^\uparrow d^\uparrow\rangle + 2|s^\downarrow d^\uparrow s^\uparrow\rangle + 2|s^\uparrow s^\downarrow d^\uparrow\rangle$$
$$+ 2|d^\uparrow s^\downarrow s^\uparrow\rangle - |s^\uparrow d^\downarrow s^\uparrow\rangle - |d^\uparrow s^\downarrow s^\uparrow\rangle - |s^\downarrow d^\uparrow s^\uparrow\rangle - |d^\downarrow s^\uparrow s^\uparrow\rangle$$
$$- |s^\uparrow s^\downarrow d^\uparrow\rangle - |s^\uparrow d^\downarrow s^\uparrow\rangle - |s^\uparrow s^\uparrow d^\downarrow\rangle - |s^\uparrow d^\uparrow s^\downarrow\rangle - |s^\downarrow s^\uparrow d^\uparrow\rangle$$
$$- |d^\downarrow s^\uparrow s^\uparrow\rangle - |s^\uparrow s^\uparrow d^\downarrow\rangle - |d^\uparrow s^\uparrow s^\downarrow\rangle \Big]$$

Combining like terms we are left with

$$= \frac{N}{\sqrt{18}} \cdot \Big[-2|s^\uparrow s^\uparrow d^\downarrow\rangle - 2|d^\downarrow s^\uparrow s^\uparrow\rangle - 2|s^\uparrow d^\downarrow s^\uparrow\rangle + |s^\uparrow d^\uparrow s^\downarrow\rangle$$
$$+ |d^\uparrow s^\uparrow s^\downarrow\rangle + |s^\downarrow s^\uparrow d^\uparrow\rangle + |s^\downarrow d^\uparrow s^\uparrow\rangle + |s^\uparrow s^\downarrow d^\uparrow\rangle + |d^\uparrow s^\downarrow s^\uparrow\rangle \Big]$$

We find that $N = \pm\sqrt{1}$ we choose $N = -1$ in order to make the wave function similar to the rest leaving us with

$$|\Xi^{-\uparrow}\rangle = \frac{1}{\sqrt{18}} \cdot \Big[2|s^\uparrow s^\uparrow d^\downarrow\rangle + 2|d^\downarrow s^\uparrow s^\uparrow\rangle + 2|s^\uparrow d^\downarrow s^\uparrow\rangle - |s^\uparrow d^\uparrow s^\downarrow\rangle$$
$$- |d^\uparrow s^\uparrow s^\downarrow\rangle - |s^\downarrow s^\uparrow d^\uparrow\rangle - |s^\downarrow d^\uparrow s^\uparrow\rangle - |s^\uparrow s^\downarrow d^\uparrow\rangle - |d^\uparrow s^\downarrow s^\uparrow\rangle \Big] \quad (40)$$

Similarly we apply $\hat{T}_+ |\Xi^{-\uparrow}\rangle = |\Xi^{0\uparrow}\rangle$ noting first

$$\hat{T}_+ |d\rangle = |u\rangle$$
$$\hat{T}_+ |u\rangle = \hat{T}_+ |s\rangle \equiv 0$$

Proceeding

$$|\Xi^{0\uparrow}\rangle = \hat{T}_+ |\Xi^{-\uparrow}\rangle = \frac{N}{\sqrt{18}} \cdot \Big[2|s^\uparrow s^\uparrow u^\downarrow\rangle + 2|u^\downarrow s^\uparrow s^\uparrow\rangle + 2|s^\uparrow u^\downarrow s^\uparrow\rangle - |s^\uparrow u^\uparrow s^\downarrow\rangle$$
$$- |u^\uparrow s^\uparrow s^\downarrow\rangle - |s^\downarrow s^\uparrow u^\uparrow\rangle - |s^\downarrow u^\uparrow s^\uparrow\rangle - |s^\uparrow s^\downarrow u^\uparrow\rangle - |u^\uparrow s^\downarrow s^\uparrow\rangle \Big] \quad (41)$$

Choosing $N = +1$ we have

$$|\Xi^{0\uparrow}\rangle = \frac{1}{\sqrt{18}} \cdot \Big[2|s^\uparrow s^\uparrow u^\downarrow\rangle + 2|u^\downarrow s^\uparrow s^\uparrow\rangle + 2|s^\uparrow u^\downarrow s^\uparrow\rangle - |s^\uparrow u^\uparrow s^\downarrow\rangle$$
$$- |u^\uparrow s^\uparrow s^\downarrow\rangle - |s^\downarrow s^\uparrow u^\uparrow\rangle - |s^\downarrow u^\uparrow s^\uparrow\rangle - |s^\uparrow s^\downarrow u^\uparrow\rangle - |u^\uparrow s^\downarrow s^\uparrow\rangle \Big] \quad (42)$$

Finally we need to construct the iso-singlet state $|\Lambda^{0\uparrow}\rangle$. We know that the origin in the Y-T_3 plane is degenerate because we do not get the same state from all directions in the multiplet. For example

$$\hat{T}_- |\Sigma^+\rangle = \hat{T}_- |T=1, T_3=1, Y=0\rangle = |T=1, T_3=0, Y=0\rangle = |\Sigma^0\rangle \quad (43)$$

Whereas

$$\begin{aligned} \hat{U}_+\hat{V}_- |\Sigma^+\rangle &= \hat{U}_+\hat{V}_- |T=1, T_3=1, Y=0\rangle = \hat{U}_+ |T=\frac{1}{2}, T_3=-\frac{1}{2}, Y=-1\rangle \\ &= |T=0, T_3=0, Y=0\rangle = |\Lambda^0\rangle \end{aligned} \quad (44)$$

$$\begin{aligned} \hat{U}_+\hat{V}_- |\Sigma^{+\uparrow}\rangle = \hat{U}_+ |\Xi^0 \uparrow\rangle = \frac{N}{\sqrt{18}} \cdot \Big[& 2\,|d^\uparrow s^\uparrow u^\downarrow\rangle + 2\,|s^\uparrow d^\uparrow u^\downarrow\rangle + 2\,|u^\downarrow d^\uparrow s^\uparrow\rangle + 2\,|u^\downarrow s^\uparrow d^\uparrow\rangle \\ & + 2\,|d^\uparrow u^\downarrow s^\uparrow\rangle + 2\,|s^\uparrow u^\downarrow d^\uparrow\rangle - |d^\uparrow u^\uparrow s^\downarrow\rangle - |s^\uparrow u^\uparrow d^\downarrow\rangle \\ & - |u^\uparrow d^\uparrow s^\downarrow\rangle - |u^\uparrow s^\uparrow d^\downarrow\rangle - |d^\downarrow s^\uparrow u^\uparrow\rangle - |s^\downarrow d^\uparrow u^\uparrow\rangle \\ & - |d^\downarrow u^\uparrow s^\uparrow\rangle - |s^\downarrow u^\uparrow d^\uparrow\rangle - |d^\uparrow s^\downarrow u^\uparrow\rangle - |s^\uparrow d^\downarrow u^\uparrow\rangle \\ & - |u^\uparrow d^\downarrow s^\uparrow\rangle - |u^\uparrow s^\downarrow d^\uparrow\rangle \Big] \end{aligned}$$

$$\begin{aligned} \hat{U}_+ |\Xi^0 \uparrow\rangle = |\tilde{\Lambda}^{0\uparrow}\rangle = \frac{1}{6} \cdot \Big[& 2\,|d^\uparrow s^\uparrow u^\downarrow\rangle + 2\,|s^\uparrow d^\uparrow u^\downarrow\rangle + 2\,|u^\downarrow d^\uparrow s^\uparrow\rangle + 2\,|u^\downarrow s^\uparrow d^\uparrow\rangle \\ & + 2\,|d^\uparrow u^\downarrow s^\uparrow\rangle + 2\,|s^\uparrow u^\downarrow d^\uparrow\rangle - |d^\uparrow u^\uparrow s^\downarrow\rangle - |s^\uparrow u^\uparrow d^\downarrow\rangle \\ & - |u^\uparrow d^\uparrow s^\downarrow\rangle - |u^\uparrow s^\uparrow d^\downarrow\rangle - |d^\downarrow s^\uparrow u^\uparrow\rangle - |s^\downarrow d^\uparrow u^\uparrow\rangle \\ & - |d^\downarrow u^\uparrow s^\uparrow\rangle - |s^\downarrow u^\uparrow d^\uparrow\rangle - |d^\uparrow s^\downarrow u^\uparrow\rangle - |s^\uparrow d^\downarrow u^\uparrow\rangle \\ & - |u^\uparrow d^\downarrow s^\uparrow\rangle - |u^\uparrow s^\downarrow d^\uparrow\rangle \Big] \end{aligned} \quad (45)$$

We demand that the Σ^0 and Λ^0 states be linearly independent. Thus we must orthogonalize the two wave functions. By orthogonalizing wave function (45) with respect to (36) by the Graham-Schmidt orthogonalization procedure we obtain the physical $|\Lambda\rangle^{0\uparrow}$ state.

Graham-Schmidt Orthogonalization

Given a linearly independent basis we can construct a orthonormal basis by

1. Scale one of the linearly independent vectors to unit length. If we let $|\mathbf{a}\rangle$ be one of our linearly independent vectors and $|\mathbf{1}\rangle, |\mathbf{2}\rangle, \cdots, |\mathbf{n}\rangle$ be our new orthonormal basis vectors
$$|\mathbf{1}\rangle = \frac{|\mathbf{a}\rangle}{\sqrt{\langle \mathbf{a}|\mathbf{a}\rangle}} \tag{46}$$

2. Subtract from the second vector $|\mathbf{b}\rangle$ is projection along the $|\mathbf{1}\rangle$
$$|\mathbf{2}'\rangle = |\mathbf{b}\rangle - |\mathbf{1}\rangle \langle \mathbf{1}|\mathbf{b}\rangle \tag{47}$$

3. Renormalize the second basis vector by its own length
$$|\mathbf{2}\rangle = \frac{|\mathbf{2}'\rangle}{\sqrt{\langle \mathbf{2}'|\mathbf{2}'\rangle}} \tag{48}$$

Now we have two orthonormal vector, this procedure can be continued indefinitely given sufficient linearly independent vectors

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Given that $|\Sigma^{0\uparrow}\rangle$ and $|\tilde{\Lambda}^{0\uparrow}\rangle$are linearly independent we can build a orthonormal basis by following the above process
$$|\Lambda^{0\uparrow}\rangle = N \cdot \Big(|\tilde{\Lambda}^{0\uparrow}\rangle - |\Sigma^{0\uparrow}\rangle \langle \Sigma^{0\uparrow}|\tilde{\Lambda}^{0\uparrow}\rangle \Big) \tag{49}$$

Thus
$$\langle \Sigma^{0\uparrow}|\Lambda^{0\uparrow}\rangle = N \cdot \Big(\langle \Sigma^{0\uparrow}|\tilde{\Lambda}^{0\uparrow}\rangle - \langle \Sigma^{0\uparrow}|\tilde{\Lambda}^{0\uparrow}\rangle \langle \Sigma^{0\uparrow}|\Sigma^{0\uparrow}\rangle \Big) = 0$$

To obtain the wave function we must first calculate $\langle \Sigma^{0\uparrow}|\tilde{\Lambda}^{0\uparrow}\rangle$
$$\begin{aligned}\langle \Sigma^{0\uparrow}|\tilde{\Lambda}^{0\uparrow}\rangle =& \frac{1}{36} \cdot \Big[2\langle d^\uparrow u^\uparrow s^\downarrow| + 2\langle u^\uparrow d^\uparrow s^\downarrow| + 2\langle s^\downarrow d^\uparrow u^\uparrow| + 2\langle s^\downarrow u^\uparrow d^\uparrow| + 2\langle d^\uparrow s^\downarrow u^\uparrow| \\ &+ 2\langle u^\uparrow s^\downarrow d^\uparrow| - \langle d^\uparrow u^\downarrow s^\uparrow| - \langle u^\uparrow d^\downarrow s^\uparrow| - \langle d^\downarrow u^\uparrow s^\uparrow| - \langle u^\downarrow d^\uparrow s^\uparrow| - \langle s^\uparrow d^\downarrow u^\uparrow| \\ &- \langle s^\uparrow u^\downarrow d^\uparrow| - \langle s^\uparrow d^\uparrow u^\downarrow| - \langle s^\uparrow u^\uparrow d^\downarrow| - \langle d^\downarrow s^\uparrow u^\uparrow| - \langle u^\downarrow s^\uparrow d^\uparrow| - \langle d^\uparrow s^\uparrow u^\downarrow| \\ &- \langle u^\uparrow s^\uparrow d^\downarrow| \Big] \Big[2|d^\uparrow s^\uparrow u^\downarrow\rangle + 2|s^\uparrow d^\uparrow u^\downarrow\rangle + 2|u^\downarrow d^\uparrow s^\uparrow\rangle + 2|u^\downarrow s^\uparrow d^\uparrow\rangle \\ &+ 2|d^\uparrow u^\downarrow s^\uparrow\rangle + 2|s^\uparrow u^\downarrow d^\uparrow\rangle - |d^\uparrow u^\uparrow s^\downarrow\rangle - |s^\uparrow u^\uparrow d^\downarrow\rangle - |u^\uparrow d^\uparrow s^\downarrow\rangle \\ &- |u^\uparrow s^\uparrow d^\downarrow\rangle - |d^\downarrow s^\uparrow u^\uparrow\rangle - |s^\downarrow d^\uparrow u^\uparrow\rangle - |d^\downarrow u^\uparrow s^\uparrow\rangle - |s^\downarrow u^\uparrow d^\uparrow\rangle \\ &- |d^\uparrow s^\downarrow u^\uparrow\rangle - |s^\uparrow d^\downarrow u^\uparrow\rangle - |u^\uparrow d^\downarrow s^\uparrow\rangle - |u^\uparrow s^\downarrow d^\uparrow\rangle \Big]\end{aligned}$$

$$= \frac{1}{36}(-2 \times 6 - 2 \times 6 + 1 \times 6) = -\frac{18}{36} = -\frac{1}{2}$$

Thus we find

$$\begin{aligned}
|\Lambda^{0\uparrow}\rangle = \frac{N}{6}\cdot\Bigg[\Bigg(&2\,|d^\uparrow s^\uparrow u^\downarrow\rangle + 2\,|s^\uparrow d^\uparrow u^\downarrow\rangle + 2\,|u^\downarrow d^\uparrow s^\uparrow\rangle + 2\,|u^\downarrow s^\uparrow d^\uparrow\rangle + 2\,|d^\uparrow u^\downarrow s^\uparrow\rangle + 2\,|s^\uparrow u^\downarrow d^\uparrow\rangle \\
&- |d^\uparrow u^\uparrow s^\downarrow\rangle - |s^\uparrow u^\uparrow d^\downarrow\rangle - |u^\uparrow d^\uparrow s^\downarrow\rangle - |u^\uparrow s^\uparrow d^\downarrow\rangle - |d^\downarrow s^\uparrow u^\uparrow\rangle - |s^\downarrow d^\uparrow u^\uparrow\rangle - |d^\downarrow u^\uparrow s^\uparrow\rangle \\
&- |s^\downarrow u^\uparrow d^\uparrow\rangle - |d^\uparrow s^\downarrow u^\uparrow\rangle - |s^\uparrow d^\downarrow u^\uparrow\rangle - |u^\uparrow d^\downarrow s^\uparrow\rangle - |u^\uparrow s^\downarrow d^\uparrow\rangle\Bigg) \\
&+ \frac{1}{2}\Bigg(\big(2\,|d^\uparrow u^\uparrow s^\downarrow\rangle + 2\,|u^\uparrow d^\uparrow s^\downarrow\rangle + 2\,|s^\downarrow d^\uparrow u^\uparrow\rangle + 2\,|s^\downarrow u^\uparrow d^\uparrow\rangle + 2\,|d^\uparrow s^\downarrow u^\uparrow\rangle \\
&+ 2\,|u^\uparrow s^\downarrow d^\uparrow\rangle - |d^\uparrow u^\downarrow s^\uparrow\rangle - |u^\uparrow d^\downarrow s^\uparrow\rangle - |d^\downarrow u^\uparrow s^\uparrow\rangle - |u^\downarrow d^\uparrow s^\uparrow\rangle - |s^\uparrow d^\downarrow u^\uparrow\rangle \\
&- |s^\uparrow u^\downarrow d^\uparrow\rangle - |s^\uparrow d^\uparrow u^\downarrow\rangle - |s^\uparrow u^\uparrow d^\downarrow\rangle - |d^\downarrow s^\uparrow u^\uparrow\rangle - |u^\downarrow s^\uparrow d^\uparrow\rangle - |d^\uparrow s^\uparrow u^\downarrow\rangle \\
&- |u^\uparrow s^\uparrow d^\downarrow\rangle\big)\Bigg)\Bigg]
\end{aligned}$$

Combing like terms,

$$\begin{aligned}
= \frac{N}{6}\cdot\Bigg[&\frac{3}{2}\Big(|d^\uparrow s^\uparrow u^\downarrow\rangle + |s^\uparrow d^\uparrow u^\downarrow\rangle + |u^\downarrow d^\uparrow s^\uparrow\rangle + |u^\downarrow s^\uparrow d^\uparrow\rangle + |d^\uparrow u^\downarrow s^\uparrow\rangle + |s^\uparrow u^\downarrow d^\uparrow\rangle\Big) \\
&- \frac{3}{2}\Big(|s^\uparrow u^\uparrow d^\downarrow\rangle + |u^\uparrow s^\uparrow d^\downarrow\rangle + |d^\downarrow s^\uparrow u^\uparrow\rangle + |d^\downarrow u^\uparrow s^\uparrow\rangle + |s^\uparrow d^\downarrow u^\uparrow\rangle + |u^\uparrow d^\downarrow s^\uparrow\rangle\Big)\Bigg]
\end{aligned}$$

With $\langle\Lambda^{0\uparrow}|\Lambda^{0\uparrow}\rangle = 1$ we find $N = \frac{2}{\sqrt{3}}$ thus finally

$$\begin{aligned}
|\Lambda^{0\uparrow}\rangle = \frac{1}{\sqrt{12}}\cdot\Bigg[&\Big(|d^\uparrow s^\uparrow u^\downarrow\rangle + |s^\uparrow d^\uparrow u^\downarrow\rangle + |u^\downarrow d^\uparrow s^\uparrow\rangle + |u^\downarrow s^\uparrow d^\uparrow\rangle + |d^\uparrow u^\downarrow s^\uparrow\rangle + |s^\uparrow u^\downarrow d^\uparrow\rangle\Big) \\
&\Big(|s^\uparrow u^\uparrow d^\downarrow\rangle + |u^\uparrow s^\uparrow d^\downarrow\rangle + |d^\downarrow s^\uparrow u^\uparrow\rangle + |d^\downarrow u^\uparrow s^\uparrow\rangle + |s^\uparrow d^\downarrow u^\uparrow\rangle + |u^\uparrow d^\downarrow s^\uparrow\rangle\Big)\Bigg]
\end{aligned} \tag{50}$$

We now have the entire set of completely symmetric, normalized octet wave functions.

Remember what we were doing?

Now that we have the wave functions for the ground states of the baryon octet we can calculate their magnetic moments simply by adding up the spin contributions from each individual quark. Let's remind ourselves of Eqn. (11) describing the magnetic moment for a hadron

$$\mu_h = \langle h| \hat{\mu} |h\rangle = \langle h| \sum_i^3 \mu_q(i)\hat{\sigma}_z(i) |h\rangle \tag{51}$$

Let's, for example, calculate the magnetic moment of the spin up proton, we have

$$\langle p^\uparrow| \hat{\mu} |p^\uparrow\rangle = \langle p^\uparrow| \sum_i^3 \mu_q(i)\hat{\sigma}_z(i) |p^\uparrow\rangle \tag{52}$$

$$\begin{aligned} = \frac{1}{18}\Bigg[\Bigg(& 4\langle u^\uparrow u^\uparrow d^\downarrow| \sum_i^3 \mu_q(i)\hat{\sigma}_z(i)|u^\uparrow u^\uparrow d^\downarrow\rangle + 4\langle d^\downarrow u^\uparrow u^\uparrow| \sum_i^3 \mu_q(i)\hat{\sigma}_z(i)|d^\downarrow u^\uparrow u^\uparrow\rangle \\ & + 4\langle u^\uparrow d^\downarrow u^\uparrow| \sum_i^3 \mu_q(i)\hat{\sigma}_z(i)|u^\uparrow d^\downarrow u^\uparrow\rangle - \langle u^\uparrow u^\downarrow d^\uparrow| \sum_i^3 \mu_q(i)\hat{\sigma}_z(i)|u^\uparrow u^\downarrow d^\uparrow\rangle \\ & + \langle u^\downarrow u^\uparrow d^\uparrow| \sum_i^3 \mu_q(i)\hat{\sigma}_z(i)|u^\downarrow u^\uparrow d^\uparrow\rangle + \langle d^\uparrow u^\downarrow u^\uparrow| \sum_i^3 \mu_q(i)\hat{\sigma}_z(i)|d^\uparrow u^\downarrow u^\uparrow\rangle \\ & + \langle d^\uparrow u^\uparrow u^\downarrow| \sum_i^3 \mu_q(i)\hat{\sigma}_z(i)|d^\uparrow u^\uparrow u^\downarrow\rangle + \langle u^\downarrow d^\uparrow u^\uparrow| \sum_i^3 \mu_q(i)\hat{\sigma}_z(i)|u^\downarrow d^\uparrow u^\uparrow\rangle \\ & + \langle u^\uparrow d^\uparrow u^\downarrow| \sum_i^3 \mu_q(i)\hat{\sigma}_z(i)|u^\uparrow d^\uparrow u^\downarrow\rangle\Bigg)\Bigg] \end{aligned} \tag{53}$$

$$\begin{aligned} =\frac{1}{18}\Bigg(& 4(2\mu_u - \mu_d) + 4(-\mu_d + 2\mu_u) + 4(-\mu_d + 2\mu_u) + (\mu_u - \mu_d - \mu_u) \\ & + (-\mu_u + \mu_u + \mu_d) + (\mu_d - \mu_u + \mu_u) + (\mu_d + \mu_u - \mu_u) + (-\mu_u + \mu_d + \mu_u) \\ & + (\mu_u + \mu_d - \mu_u)\Bigg) \end{aligned} \tag{54}$$

$$\mu_p = \frac{16}{18}\mu_u - \frac{6}{18}\mu_d = \frac{1}{3}(4\mu_u - \mu_d) \tag{55}$$

Similarly if we look at the neutron wave function in Eq (33) we immediately see that the total magnetic moment will be the same with μ_u and μ_d interchanged

$$\mu_n = \frac{1}{3}(4\mu_d - \mu_u) \tag{56}$$

For Σ^+ we have

$$\mu_{\Sigma^+} = \langle \Sigma^+ | \sum_i^3 \mu_q(i) \hat{\sigma}_z(i) | \Sigma^+ \rangle \tag{57}$$

$$= \frac{1}{18} \Big(4(2\mu_u - \mu_s) + 4(-\mu_s + 2\mu_u) + 4(\mu_u - \mu_s + \mu_u) + (\mu_u - \mu_u + \mu_s) + (-\mu_u + \mu_u + \mu_s) + (\mu_s - \mu_u + \mu_u) + (\mu_s + \mu_u - \mu_u) + (-\mu_u + \mu_s + \mu_u) + (\mu_u + \mu_s - \mu_u) \Big) \tag{58}$$

$$= \frac{1}{18}(24\mu_u - 6\mu_s) = \frac{4}{3}\mu_u - \frac{1}{3}\mu_s \tag{59}$$

For Σ^0 we can see from Eqn. (36)

$$\begin{aligned}\mu_{\Sigma^0} = \frac{1}{36}\big(&4(\mu_d + \mu_u - \mu_s) + 4(\mu_u + \mu_d - \mu_s) + 4(-\mu_s + \mu_d + \mu_u) + 4(-\mu_s + \mu_u + \mu_d) \\ &+ 4(\mu_d - \mu_s + \mu_u) + 4(\mu_u - \mu_s + \mu_d) + (\mu_d - \mu_u + \mu_s) + (\mu_u - \mu_d + \mu_s) \\ &+ (-\mu_d + \mu_u + \mu_s) + (-\mu_u + \mu_d + \mu_s) + (\mu_s - \mu_d + \mu_u) + (\mu_s - \mu_u + \mu_d) \\ &+ (\mu_s + \mu_d - \mu_u) + (\mu_s + \mu_u - \mu_d) + (-\mu_d + \mu_s + \mu_u) + (-\mu_u + \mu_s + \mu_d) \\ &+ (\mu_d + \mu_s - \mu_u) + (\mu_u + \mu_s - \mu_d)\big)\end{aligned}$$

Combining like-terms

$$= \frac{1}{36}(24\mu_d - 12\mu_s + 24\mu_u) = \frac{2}{3}(\mu_d + \mu_u) - \frac{1}{3}\mu_s \tag{60}$$

For the Σ^- we notice immediately from the wave functions (Eqn. (34)) that we can interchange μ_u and μ_d within μ_{Σ^+} to obtain μ_{Σ^-}

$$\mu_{\Sigma^-} = \frac{4}{3}\mu_d - \frac{1}{3}\mu_s \tag{61}$$

Likewise for μ_{Ξ^-} we interchange $\mu_d \to \mu_s$ and $\mu_s \to \mu_d$ within μ_{Σ^-}

$$\mu_{\Xi^-} = \frac{4}{3}\mu_s - \frac{1}{3}\mu_d \tag{62}$$

Similarly to obtain μ_{Ξ^0} we interchange $\mu_d \to \mu_u$ in μ_{Ξ^-}

$$\mu_{\Xi^0} = \frac{4}{3}\mu_s - \frac{1}{3}\mu_u \tag{63}$$

For μ_{Λ^0} we have

$$\mu_{\Lambda^0} = \frac{1}{12}\big((\mu_d + \mu_s - \mu_u) + (\mu_s + \mu_d - \mu_u) + (-\mu_u + \mu_d + \mu_s) + (-\mu_u + \mu_s + \mu_d)$$
$$+ (\mu_d - \mu_u + \mu_s) + (\mu_s - \mu_u + \mu_d) + (\mu_s + \mu_u - \mu_d) + (\mu_u + \mu_s - \mu_d)$$
$$+ (-\mu_d + \mu_s + \mu_u) + (-\mu_d + \mu_u + \mu_s) + (\mu_s - \mu_d + \mu_u) + (\mu_u - \mu_d + \mu_s)\big)$$

$$\mu_{\Lambda^0} = \mu_s \tag{64}$$

Now that we have our theoretical expressions we can begin to compare with actual experimental data. We have many expressions relating the magnetic moments of baryons from the octet μ_h to individual magnetic moments of the constituent quarks μ_u, μ_d, and μ_s. Thus, we need to obtain the predicted values of individual quark moments by comparing with experiment. We can obtain a prediction of μ_u and μ_d by comparing experimental values of μ_p and μ_n

$$\frac{\mu_p}{\mu_n} = \frac{\frac{1}{3}(4\mu_u - \mu_d)}{\frac{1}{3}(4\mu_d - \mu_u)} \tag{65}$$

If we assume that a quark is a point-like particle like the electron, then its magnetic moment can be expressed approximately as

$$\mu_q = \frac{e}{2m_q} Q\vec{\sigma} \tag{66}$$

Where eQ is the quark charge, m_q is the mass of the quark, and $\vec{\sigma}$ are the Pauli-spin matrices. If we ignore mass differences between the quarks then the magnetic moments are proportional to the individual quark charges. From this we can obtain a relation between the magnetic moments of the up and down quarks.

$$\mu_u = \frac{Q_u}{Q_d}\mu_d = \frac{2/3}{-1/3}\mu_d = -2\mu_d \tag{67}$$

Plugging this result back into Eq(63)

$$\frac{\mu_p}{\mu_n} = \frac{-8\mu_d - \mu_d}{4\mu_d + 2\mu_d} = -\frac{3}{2} = -1.5 \tag{68}$$

Experimentally $\mu_p = 2.7928473446 \pm 0.0000000008$ and $\mu_n = -1.9130427 \pm 0.0000005$ thus $\frac{\mu_p}{\mu_n} = -1.4598980 \pm 0.00000005$. The agreement between theory and experiment is striking. From experimental values of μ_p and μ_n we can calculate the individual quark moments.

$$\mu_u = \frac{1}{5}\mu_n + \frac{4}{5}\mu_p = 1.852\mu_0 \tag{69}$$

$$\mu_d = \frac{1}{5}\mu_u + \frac{4}{5}\mu_n = -0.972\mu_0 \tag{70}$$

$$\mu_s = \mu_{\Lambda^0} = (-0.613 \pm 0.004)\mu_0 \tag{71}$$

$$\mu_0 = \frac{e}{2m_q} \tag{72}$$

Using these value we can obtain predictions for the megnetic moments for the rest of the baryon octet. This is summarized in the table below

Baryon	μ_h	Prediction $[\mu_0]$	Experiment $[\mu_0]$
Σ^+	$\frac{4}{3}\mu_u - \frac{1}{3}\mu_s$	2.673	2.458 $\pm$ 0.010
Σ^0	$\frac{2}{3}(\mu_d + \mu_u) - \frac{1}{3}\mu_s$	0.791	-
Σ^-	$\frac{4}{3}\mu_d - \frac{1}{3}\mu_s$	-1.091	-1.160 $\pm$ 0.025
Ξ^-	$\frac{4}{3}\mu_s - \frac{1}{3}\mu_d$	-0.493	-0.6507 $\pm$ 0.0025
Ξ^0	$\frac{4}{3}\mu_s - \frac{1}{3}\mu_u$	-1.435	-1.250 $\pm$ 0.014

Theory and experiment agree surprisingly well for this rudimentary additive quark model. The discrepancies above can be attributed to d-wave contributions to the orbital angular momentum.

PART II: $\Sigma - \Lambda$ Mass Difference

If SU(6) symmetry were an exact symmetry of the strong interactions we would expect all the energies within a given SU(6) multiplet to be degenerate (have the same mass). We observe from experiment, however, non-degenerate energies for a given SU(6) multiplet (For example the 56-plet has a mass splitting of order 10%). We can imagine that this is due to a small symmetry breaking term in the Hamiltonian. From this realization we construct the Hamiltonian with two parts $\hat{H}_0$ which is invariant under SU(6) transformations and $\hat{H}_b$ which breaks the SU(6) symmetry. Because the energy splitting is small we can assume that the mass contributed by $\hat{H}_b$ is small compared to $\hat{H}_0$

$$\hat{H} = \hat{H}_0 + \hat{H}_b$$

$$M = \langle \hat{H}_0 \rangle + \langle \hat{H}_b \rangle \,, \langle \hat{H}_0 \rangle \gg \langle \hat{H}_b \rangle$$

From SU(3) we were able to obtain a mass formula by first ignoring the electromagnetic mass splittings and assuming isospin was a good symmetry (particles in the same isoplet have the same mass). This meant that the symmetry breaking Hamiltonian had to commute with isospin $[\hat{H}_b, \hat{T}_3] = 0$. The only generator in SU(3) which commutes with $\hat{T}_3$ is $\hat{\lambda}_8 = 2\hat{F}_8 = \sqrt{3}\hat{Y}$ thus the simplest assumption is that the symmetry is broken by hypercharge and we get

$$\hat{H}_b = b\hat{Y} \tag{73}$$

Calculating the expectation value of $\hat{H}_b$ on an unperturbed wave function, first order perturbation theory gives us

$$M = a + bY \tag{74}$$

Where $a = M_0$ when $Y = 0$ and a and b are constants. This mass formula produces satisfactory mass relations within the baryon resonances but fails for the baryon octet. The reason is $Y_{\Sigma^0} = Y_{\Lambda^0}$ which yields the mass relation $M_{\Sigma^0} = M_{\Lambda^0}$ when in reality $M_{\Sigma^0} - M_{\Lambda^0} \approx 77$MeV. To account for this we modify $\hat{H}_b$ by adding $\hat{\mathbf{T}}^2$ and $\hat{Y}^2$ (the only other operators available operators within SU(3) which commute with $\hat{T}_3$).

$$\hat{H}_b = b\hat{Y} + c\hat{T}^2 + d\hat{Y}^2 \tag{75}$$

Which gives a mass formula of

$$M = a + bY + cT(T+1) + dY^2 \tag{76}$$

Where c and d are constants in an SU(3) multiplet. We have to put constraints on this mass formula in order to produce constant mass splitting in the decuplet and after doing so we end up with the Gell-Mann Okubo mass formula

$$M = a + bY + c\big[T(T+1) - \frac{1}{4}Y^2\big] \tag{77}$$

When incorporating SU(6) symmetry we say that the symmetry breaking Hamiltonian is scalar in spin space i.e. $[\hat{H}_b, \hat{J}^2] = 0$ [3]. Thus, $\hat{H}_b \propto \hat{J}^2$ we obtain the following Gürsey-Radicati mass formula.

$$M = a + bY + c\big[T(T+1) - \frac{1}{4}Y^2\big] + d\big[J(J+1)\big] \tag{78}$$

Where d is a constant and has the same value in different spin multiplets. Using experimental mass data I can fit the parameters a, b, c, d and then use those parameters to calculate the masses of Σ^0 and Λ^0

$$M_N[939\text{MeV}] = a + b + \frac{1}{2}c + \frac{3}{4}d$$

[3]We use $\hat{\mathbf{J}}^2$ because it commutes with $\hat{T}_3$

$$M_{\Sigma^*}[1384\text{MeV}] = a + 2c + \frac{15}{4}d$$
$$M_{\Xi}[1315\text{MeV}] = a - b + \frac{1}{2}c + \frac{3}{4}d$$
$$M_{\Xi^*}[1531\text{ MeV}] = a - b + \frac{1}{2}c + \frac{15}{4}d$$

From the fit

$$a = 1059.33\text{ MeV}$$
$$b = -188\text{ MeV}$$
$$c = 27.33\text{ MeV}$$
$$d = 72\text{ MeV}$$

For M_Σ and M_Λ we have

$$M_\Sigma = a + 2c + \frac{3}{4}d$$
$$M_\Lambda = a + \frac{3}{4}d$$

Plugging in the parameters we find

$$M_\Sigma = 1168\text{ MeV} \tag{79}$$

$$M_\Lambda = 1113\text{ MeV} \tag{80}$$

Thus,

$$M_\Sigma - M_\Lambda = 55\text{ MeV} \tag{81}$$

Experimentally the mass difference is

$$\begin{aligned} M_{\Sigma,\text{Exp}} - M_{\Lambda,\text{Exp}} &= 1192.642 \pm 0.024\text{ MeV} - 1115.683 \pm 0.006\text{ MeV} \\ &= 76.96 \pm 0.02\text{ MeV} \end{aligned} \tag{82}$$

The theory and experiment are of the same order of magnitude. As mentioned by Radiciati and Gürsey, the mass formula they proposed is not the most general formula. In the next section I will outline the procedure to obtain mass differences which agree exactly(ground state) with experiment by using the generalized Gürsey-Radicati mass formula.

Generalized Gursey-Radicati mass formula

The mass formula obtained in Eqn. (78) can be rewritten in terms of its Casimir operators

$$M = a + bC_1[U_Y(1)] + c\left[C_2[SU_I(2)] - \frac{1}{4}\big(C_1[U_Y(1)]\big)^2\right] + dC_2[SU_S(2)] \tag{83}$$

Where $C_2[SU_I(2)]$ and $C_2[SU_S(2)]$ are the (quadratic) Casimir operators for isospin and spin and $C_1[U_Y(1)]$ is the Casimir operator for the U(1) subgroup generated by hypercharge (Y). To generalize the above formula we can think of a dynamical spin-flavor symmetry $SU_{\mathrm{sf}}(6)$ with subgroups

$$SU_{\mathrm{sf}}(6) \supset SU_{\mathrm{f}}(3) \otimes SU_{\mathrm{s}}(2) \supset SU_{\mathrm{I}}(2) \otimes U_{\mathrm{Y}}(1) \otimes SO_{\mathrm{s}}(2) \tag{84}$$

With quantum numbers $(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_5)$, (λ_f, μ_f), J, I, Y, and J_z which define the irreducible representations of each group. With this we can rewrite the generalized symmetry breaking mass formula

$$\begin{aligned} M =& a + bC_2[SU_{\mathrm{sf}}(6)] + cC_2[SU_{\mathrm{f}}(3)] + dC_2[SU_{\mathrm{s}}(2)] + eC_1[U_Y(1)] \\ &+ f\left[C_2[SU_I(2)] - \frac{1}{4}\big(C_1[U_Y(1)]\big)^2\right] \end{aligned} \tag{85}$$

In general an $SU_{\mathrm{sf}}(6)$ wavefunction can be expressed in more than one way. We could label the wave function with the quantum numbers given by the group and its subgroups

$$|(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5), (\lambda_f, \mu_f), J, I, Y, J_z\rangle$$

Often the representations are represented by their dimensions

$$|^{2S+1}\mathrm{dim}\big(SU(3)\big)_J, [\mathrm{dim}\big(SU(6)\big), L^P], X\rangle \tag{86}$$

where dim(SU(n)) is the dimension of the representation, S and L are the total spin and orbital angular momentum, J and P are the spin and parity of the resonance and X denotes the baryon. For example the nucleon wave function in the ground state can be denoted as follows

$$\begin{aligned} |N\rangle &= |(3,0,0,0,0),(2,1),\frac{1}{2},1,\frac{1}{2},J_z\rangle \\ &= |^2 8_{\frac{1}{2}}, [56, 0^+], N\rangle \end{aligned} \tag{87}$$

Either notation uniquely represents the state. Just as we did before in order to find the symmetry breaking energy contribution we need the expectation value of H_b on the eigenfunctions of the invariant Hamiltonian.

$$\hat{H}_b = bC_2[SU_{\mathrm{sf}}(6)]+cC_2[SU_{\mathrm{f}}(3)]+dC_2[SU_{\mathrm{s}}(2)]+eC_1[U_Y(1)]+f\left[C_2[SU_I(2)]-\frac{1}{4}\big(C_1[U_Y(1)]\big)^2\right] \tag{88}$$

First order perturbation expansion yields

$$\langle(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5),(\lambda_f,\mu_f),J,I,Y,J_z|\hat{H}_b|(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5),(\lambda_f,\mu_f),J,I,Y,J_z\rangle \tag{89}$$

$$= b\langle C_2[SU_{\mathrm{sf}}(6)]\rangle + c\langle C_2[SU_{\mathrm{f}}(3)]\rangle + dJ(J+1) + eY + f\big[T(T+1) - \frac{1}{4}Y^2\big] \tag{90}$$

Where

$$\langle C_2[SU_{\mathrm{sf}}(6)]\rangle = \begin{cases} 45/4 \text{ for } [56] \\ 33/3 \text{ for } [70] \\ 21/4 \text{ for } [20] \end{cases}$$

$$\langle C_2[SU_{\mathrm{f}}(3)]\rangle = \begin{cases} 3 \text{ for } [8] \\ 6 \text{ for } [10] \\ 0 \text{ for } [1] \end{cases}$$

Thus leaving us with two mass formulas, one for the light baryons (octet) and another for the baryon resonance (decuplet)

$$M_{[8]} = a + \frac{45}{4}b + 3c + dJ(J+1) + eY + f\big[T(T+1) - \frac{1}{4}Y^2\big] \tag{91}$$

$$M_{[10]} = a + \frac{45}{4}b + 6c + dJ(J+1) + eY + f\big[T(T+1) - \frac{1}{4}Y^2\big] \tag{92}$$

There are two conditions which need to be valid in order to use the generalized Gürsey Radicati mass formula. The first is that it must be valid to use the same fitting parameters between different multiplets as seen above. The second condition is that you must have a reliable way of determining the mean mass of the multiplet (a). This is accomplished by solving the Schrodinger equation (non-relativistic) and obtaining reliable eigenfunctions for a given invariant Hamiltonian. The eigenvalues of these eigenfunctions will give you the values for M_0 in a given multiplet. One successful and popular technique is to solve for eigenfunctions subject to a hypercentral potential. The three-body quark wave functions obtained make up the hypercentral constiuent quark model (hCQM). This model describes baryons as three-quark bound states. Internal quark motion is defined using Jacobi coordinates

$$\vec{\rho} = \frac{1}{\sqrt{2}}(\vec{r}_1 - \vec{r}_2) \tag{93}$$

$$\vec{\lambda} = \frac{1}{\sqrt{6}}(\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3) \tag{94}$$

With hyperradius

$$x = \sqrt{|\rho|^2 + |\lambda|^2} \tag{95}$$

Consider the invariant three body Hamiltonian

$$H_0 = 3m + \frac{\mathbf{p}_\rho^2}{2m} + \frac{\mathbf{p}_\lambda^2}{2m} + V(x) \tag{96}$$

Where m is the quark mass[4], and $V(x)$ is the hypercentral potential

$$V(x) = -\frac{\tau}{x} + \alpha x \tag{97}$$

Where τ and α are fitting parameters. Given this Hamiltonian we can solve the Schrodinger equation

$$\frac{\nabla^2}{2m}\psi_{\gamma\nu} = (E_{\gamma\nu} - V(x))\psi_{\gamma\nu} \tag{98}$$

Where ∇^2 in hyperspherical coordinates is

$$\nabla^2 = (\frac{d}{dx^2} + \frac{5}{x}\frac{d}{dx} + \frac{\gamma(\gamma+4)}{x^2}) \tag{99}$$

This model not only makes predictions for particles in the ground state but also excited states with orbital angular momentum. For example, Giannini et al. [5] fit the constants given in Eqn. (88) using three and four star resonances to obtain the symmetry breaking contribution to the baryon masses. These contributions along with there calculations of the average mass of a given multiplet from the hQCM result in ground state predictions which are essentially indistinguishable from experiment. From [5] they calculated Σ and Λ masses

$$M_\Sigma^{calc[5]} = 1193 \text{ MeV}$$

$$M_\Lambda^{calc[5]} = 1116 \text{ MeV}$$

$$M_\Sigma^{calc[5]} - M_\Lambda^{calc[5]} = 77 \text{ MeV} \tag{100}$$

[4]This particular hCQM neglects the difference in quark masses for simplicity

Calculating $G_A(q^2 = 0)$ [2]

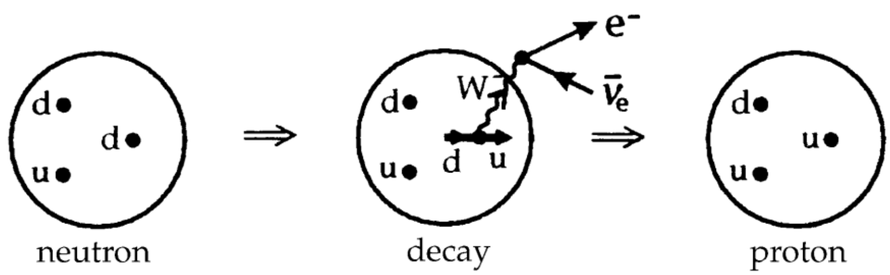

Figure 5: Weak neutron beta decay [2]

In the case of weak interactions we are interested in calculating the axial vector form factor $G(q^2)$ at $q^2 = 0$. This corresponds to the weak beta decay

$$n \to p + e^- + \bar{\nu}_e \tag{101}$$

This process includes the conversion of d quark inside the neutron to a u quark with the emission of an electron and an anti-electron neutrino. The charged current which is responsible for this process

$$J_\mu^{(+)}(d \to u) = \cos(\theta_c)[\bar{u}\gamma_\mu(1-\gamma_5)d] \tag{102}$$

Where θ_c is the Cabibo mixing angle. Ignoring the cos θ_c the transition operator for a single d quark is of the form

$$\gamma_\mu(1-\gamma_5)\hat{T}_+ \tag{103}$$

Where, as we know from calculating the baryon wave functions, transforms the down quark into an up quark. Generalizing as a many-body operator we have

$$\sum_{i=1}^{3} \gamma_{(i)}^{\mu}(1-\gamma_5^{(i)})\hat{T}_+^{(i)} \tag{104}$$

In the non-relativistic case the only terms which contribute to the transition are the spatial components of the axial vector

$$g_A = \langle p^\uparrow|\sigma_3^{(i)}\hat{T}_+^{(i)}|n^\uparrow\rangle \tag{105}$$

Let's rewrite the neutron and proton wave functions in a more appealing form for the calculation. From Eqn. (22) we had

$$\begin{aligned} |p^{\uparrow}\rangle =& \frac{1}{\sqrt{18}}\Big[2\,|u^{\uparrow}u^{\uparrow}d^{\downarrow}\rangle + 2\,|d^{\downarrow}u^{\uparrow}u^{\uparrow}\rangle + 2\,|u^{\uparrow}d^{\downarrow}u^{\uparrow}\rangle - |u^{\uparrow}u^{\downarrow}d^{\uparrow}\rangle \\ & - |u^{\downarrow}u^{\uparrow}d^{\uparrow}\rangle - |d^{\uparrow}u^{\downarrow}u^{\uparrow} - |d^{\uparrow}u^{\uparrow}u^{\downarrow}\rangle - |u^{\downarrow}d^{\uparrow}u^{\uparrow}\rangle - u^{\uparrow}d^{\uparrow}u^{\downarrow}\rangle\Big] \end{aligned} \tag{106}$$

Rewriting in factorized form

$$\begin{aligned} |p^{\uparrow}\rangle = & \frac{1}{6\sqrt{2}}\Big(|udu\rangle + |duu\rangle - 2\,|uud\rangle\Big)\Big(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle - 2\,|\uparrow\uparrow\downarrow\rangle\Big) \\ & + \frac{1}{2\sqrt{2}}\Big(|udu\rangle - |duu\rangle\Big)\Big(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\Big) \end{aligned} \tag{107}$$

From Eqn. (31) we have the neutron wave function

$$\begin{aligned} |n^{\uparrow}\rangle =& \frac{1}{\sqrt{18}}\Big[|d^{\uparrow}u^{\uparrow}d^{\downarrow}\rangle + |u^{\uparrow}d^{\uparrow}d^{\downarrow}\rangle + |d^{\downarrow}d^{\uparrow}u^{\uparrow}\rangle + |d^{\downarrow}u^{\uparrow}d^{\uparrow}\rangle \\ & + |d^{\uparrow}d^{\downarrow}u^{\uparrow}\rangle + |u^{\uparrow}d^{\downarrow}d^{\uparrow}\rangle - 2\,|d^{\uparrow}u^{\downarrow}d^{\uparrow}\rangle - 2\,|u^{\downarrow}d^{\uparrow}d^{\uparrow}\rangle - 2\,|d^{\uparrow}d^{\uparrow}u^{\downarrow}\rangle\Big] \end{aligned} \tag{108}$$

Rewriting

$$\begin{aligned} |n^{\uparrow}\rangle = & -\frac{1}{6\sqrt{2}}\Big(|udd\rangle + |dud\rangle - 2\,|ddu\rangle\Big)\Big(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle - 2\,|\uparrow\uparrow\downarrow\rangle\Big) \\ & + \frac{1}{2\sqrt{2}}\Big(|udd\rangle - |dud\rangle\Big)\Big(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\Big) \end{aligned} \tag{109}$$

First calculating $\hat{T}_+$ on the neutron wave function

$$\begin{aligned} \hat{T}_+\,|n^{\uparrow}\rangle = & -\frac{1}{6\sqrt{2}}\Big(|uud\rangle + |udu\rangle + |uud\rangle + |duu\rangle - 2\,|udu\rangle - 2\,|duu\rangle\Big)\Big(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle - 2\,|\uparrow\uparrow\downarrow\rangle\Big) \\ & + \frac{1}{2\sqrt{2}}\Big(|uud\rangle + |udu\rangle - |uud\rangle - |duu\rangle\Big)\Big(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\Big) \end{aligned} \tag{110}$$

$$\begin{aligned} = & -\frac{1}{6\sqrt{2}}\Big(2\,|uud\rangle - |udu\rangle - |duu\rangle\Big)\Big(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle - 2\,|\uparrow\uparrow\downarrow\rangle\Big) \\ & + \frac{1}{2\sqrt{2}}\Big(|udu\rangle - |duu\rangle\Big)\Big(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\Big) \end{aligned} \tag{111}$$

$$= |p^{\uparrow}\rangle \tag{112}$$

Now evaluating $\sum_{i=1}^{3} \sigma_3 \hat{T}_+ |n^{\uparrow}\rangle$

$$\begin{aligned}
\sum_{i=1}^{3} \sigma_3 \hat{T}_+ |n^{\uparrow}\rangle = &- \frac{1}{6\sqrt{2}}\Big(|uud\rangle - 2|udu\rangle\Big)\Big(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle - 2|\uparrow\uparrow\downarrow\rangle\Big) \\
&+ \frac{1}{2\sqrt{2}}\Big(-|uud\rangle\Big)\Big(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle\Big) \\
&- \frac{1}{6\sqrt{2}}\Big(|uud\rangle - 2|duu\rangle\Big)\Big(-|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle - 2|\uparrow\uparrow\downarrow\rangle\Big) \\
&+ \frac{1}{2\sqrt{2}}\Big(|uud\rangle\Big)\Big(-|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\Big) \\
&- \frac{1}{6\sqrt{2}}\Big(|udu\rangle + |duu\rangle\Big)\Big(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle + 2|\uparrow\uparrow\downarrow\rangle\Big) \\
&+ \frac{1}{2\sqrt{2}}\Big(|udu\rangle - |duu\rangle\Big)\Big(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\Big)
\end{aligned} \tag{113}$$

$$\begin{aligned}
= &- \frac{1}{6\sqrt{2}}\Big[\Big(-4|uud\rangle|\uparrow\uparrow\downarrow\rangle\Big) + \Big(|udu\rangle\big(-|\uparrow\downarrow\uparrow\rangle + 3|\downarrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle\big)\Big) \\
&+ \Big(|duu\rangle\big(3|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle + 6|\uparrow\uparrow\downarrow\rangle\big)\Big)\Big] - \frac{1}{2\sqrt{2}}\Big[\Big(2|uud\rangle\big(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle\big)\Big) \\
&- \Big(\big(|udu\rangle - |duu\rangle\big)\big(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\big)\Big)\Big]
\end{aligned} \tag{114}$$

$$\begin{aligned}
= &- \frac{1}{6\sqrt{2}}\Big[\Big(2|uud\rangle\big(3|\uparrow\downarrow\uparrow\rangle + 3|\downarrow\uparrow\uparrow\rangle - 2|\uparrow\uparrow\downarrow\rangle\big)\Big) \\
&+ \Big(|udu\rangle\big(-|\uparrow\downarrow\uparrow\rangle + 3|\downarrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle\big)\Big) \\
&+ \Big(|duu\rangle\big(3|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle + 6|\uparrow\uparrow\downarrow\rangle\big)\Big)\Big] \\
&+ \frac{1}{2\sqrt{2}}\Big[\Big(\big(|udu\rangle - |duu\rangle\big)\big(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\big)\Big)\Big]
\end{aligned} \tag{115}$$

And calculating $\langle p^{\uparrow}| \sum_{i=1}^{3} \sigma_3 \hat{T}_+ |n^{\uparrow}\rangle$

$$g_A = \langle p^{\uparrow}| \sum_{i=1}^{3} \sigma_3 \hat{T}_+ |n^{\uparrow}\rangle = \frac{60}{72} + \frac{1}{3} + \frac{1}{2} = \frac{5}{3} \tag{116}$$

The experimental measure of g_A

$$g_A^{exp} = -1.2724 \pm 0.0023 \tag{117}$$

$$g_A^{theory} = -1.67 \tag{118}$$

This is a large discrepancy and one of the many shortcomings of the SU(6) quark model. As discussed by Gürsey and Radicati [7] this value is actually the unrenormalized value for the axial vector constant, see [8].

Concluding Remarks

We have now seen three predictive calculations in the SU(6) quark model. Somewhat surprisingly the model makes very agreeable predictions in regards to the baryon octet magnetic moments. We saw that the additive quark model made decent predictions for the mass splittings within the multiplets. Of course, the hCQM and generalized GR mass formula made very accurate predictions of the ground state and excited state baryons. Finally in the last calculation of the axial vector constant, SU(6) did not produce an accurate result. This was a clue that a more accurate theory was needed to describe the weak interactions. Of course probably the most glaring issue with SU(6) is that the calculated ground state wave functions are totally symmetric under interchange of two quarks. We know that fermion wave functions must be totally anti-symmetric. So we know SU(6) can't be the end of the story, bring on the color index!

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1 Appendix

Below are the generators for SU(6)

$$\lambda_1 = \begin{bmatrix} \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & (0)_{3\times3} \\ (0)_{3\times3} & (0)_{3\times3} \end{bmatrix}, \lambda_2 = \begin{bmatrix} \begin{matrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & (0)_{3\times3} \\ (0)_{3\times3} & (0)_{3\times3} \end{bmatrix}$$

$$\lambda_3 = \begin{bmatrix} \begin{matrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{matrix} & (0)_{3\times3} \\ (0)_{3\times3} & (0)_{3\times3} \end{bmatrix}, \lambda_4 = \begin{bmatrix} \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix} & (0)_{3\times3} \\ (0)_{3\times3} & (0)_{3\times3} \end{bmatrix}$$

$$\lambda_5 = \begin{bmatrix} \begin{matrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{matrix} & (0)_{3\times3} \\ (0)_{3\times3} & (0)_{3\times3} \end{bmatrix}, \lambda_6 = \begin{bmatrix} \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix} & (0)_{3\times3} \\ (0)_{3\times3} & (0)_{3\times3} \end{bmatrix}$$

$$\lambda_7 = \begin{bmatrix} \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{matrix} & (0)_{3\times3} \\ (0)_{3\times3} & (0)_{3\times3} \end{bmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{matrix} & (0)_{3\times3} \\ (0)_{3\times3} & (0)_{3\times3} \end{bmatrix}$$

$$\lambda_9 = \begin{bmatrix} (0)_{3\times3} & \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & (0)_{3\times3} \end{bmatrix} \lambda_{10} = \begin{bmatrix} (0)_{3\times3} & \begin{matrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & (0)_{3\times3} \end{bmatrix}$$

$$\lambda_{11} = \begin{bmatrix} (0)_{3\times3} & \begin{matrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & (0)_{3\times3} \end{bmatrix} \lambda_{12} = \begin{bmatrix} (0)_{3\times3} & \begin{matrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & (0)_{3\times3} \end{bmatrix}$$

$$\lambda_{13} = \begin{bmatrix} (0)_{3\times3} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & (0)_{3\times3} \end{bmatrix} \lambda_{14} = \begin{bmatrix} (0)_{3\times3} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & (0)_{3\times3} \end{bmatrix}$$

$$\lambda_{15} = \begin{bmatrix} (0)_{3\times3} & \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & (0)_{3\times3} \end{bmatrix} \lambda_{16} = \begin{bmatrix} (0)_{3\times3} & \begin{matrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & (0)_{3\times3} \end{bmatrix}$$

$$\lambda_{17} = \begin{bmatrix} (0)_{3\times3} & \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} & (0)_{3\times3} \end{bmatrix} \lambda_{18} = \begin{bmatrix} (0)_{3\times3} & \begin{matrix} 0 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{matrix} & (0)_{3\times3} \end{bmatrix}$$

$$\lambda_{19} = \begin{bmatrix} (0)_{3\times3} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} & (0)_{3\times3} \end{bmatrix} \lambda_{20} = \begin{bmatrix} (0)_{3\times3} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -i & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{matrix} & (0)_{3\times3} \end{bmatrix}$$

$$\lambda_{21} = \begin{bmatrix} (0)_{3\times3} & \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix} & (0)_{3\times3} \end{bmatrix} \lambda_{22} = \begin{bmatrix} (0)_{3\times3} & \begin{matrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{matrix} & (0)_{3\times3} \end{bmatrix}$$

$$\lambda_{23} = \begin{bmatrix} & (0)_{3\times3} & & 0 & 0 & 0 \\ & & & 0 & 0 & 1 \\ & & & 0 & 0 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & (0)_{3\times3} & \\ 0 & 1 & 0 & & & \end{bmatrix} \lambda_{24} = \begin{bmatrix} & (0)_{3\times3} & & 0 & 0 & 0 \\ & & & 0 & 0 & -i \\ & & & 0 & 0 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & (0)_{3\times3} & \\ 0 & i & 0 & & & \end{bmatrix}$$

$$\lambda_{25} = \begin{bmatrix} & (0)_{3\times3} & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 1 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & (0)_{3\times3} & \\ 0 & 0 & 1 & & & \end{bmatrix} \lambda_{26} = \begin{bmatrix} & (0)_{3\times3} & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & -i \\ 0 & i & 0 & & & \\ 0 & 0 & 0 & & (0)_{3\times3} & \\ 0 & 0 & i & & & \end{bmatrix}$$

$$\lambda_{27} = \begin{bmatrix} (0)_{3\times3} & (0)_{3\times3} \\ (0)_{3\times3} & \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \end{bmatrix} \lambda_{28} = \begin{bmatrix} (0)_{3\times3} & (0)_{3\times3} \\ (0)_{3\times3} & \begin{matrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \end{bmatrix}$$

$$\lambda_{29} = \begin{bmatrix} (0)_{3\times3} & (0)_{3\times3} \\ (0)_{3\times3} & \begin{matrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{matrix} \end{bmatrix} \lambda_{30} = \begin{bmatrix} (0)_{3\times3} & (0)_{3\times3} \\ (0)_{3\times3} & \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix} \end{bmatrix}$$

$$\lambda_{31} = \begin{bmatrix} (0)_{3\times3} & (0)_{3\times3} \\ (0)_{3\times3} & \begin{matrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{matrix} \end{bmatrix} \lambda_{32} = \begin{bmatrix} (0)_{3\times3} & (0)_{3\times3} \\ (0)_{3\times3} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix} \end{bmatrix}$$

$$\lambda_{33} = \begin{bmatrix} (0)_{3\times3} & (0)_{3\times3} \\ (0)_{3\times3} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{matrix} \end{bmatrix} \lambda_{34} = \frac{1}{\sqrt{3}} \begin{bmatrix} (0)_{3\times3} & (0)_{3\times3} \\ (0)_{3\times3} & \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{matrix} \end{bmatrix}$$

$$\lambda_{35} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 & 0 & 0 & & & \\ 0 & -1 & 0 & & (0)_{3\times3} & \\ 0 & 0 & -1 & & & \\ & & & 1 & 0 & 0 \\ & (0)_{3\times3} & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \end{bmatrix}$$