RANDALL-SUNDRUM MODELS

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1 A bit of General Relativity

In general relativity the metric is promoted to a dynamical field variable. In the presence of an energy density, the metric (spacetime) acquires off diagonal elements resulting in curvature. We define curvature in terms of the parallel transport of vectors along closed curves. In the presence of curvature, derivative operators do not commute and upon travelling along a closed curve we find that the initial vector V does not remain parallel to the parallel-transported vector V'. When we speak of parallel transport we mean that the vector is moved along the instantaneous tangent plane to the surface along the curve.

We define the (covariant) derivative operator ∇, on a manifold M as a map which takes each smooth tensor field of type (k, l) to a smooth tensor field of type $(k, l+1)$. That is, the derivative operator tacks on one more covariant index to the tensor field with which it acts upon. The derivative operator also satisfies five properties

1. Linearity

$$\nabla_c \left(\alpha A^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} + \beta B^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}\right) = \alpha \nabla_c A^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} + \beta \nabla_c B^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} \tag{1}$$

2. Leibnitz rule

$$\nabla_e \left[A^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} B^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}\right] = \left[\nabla_e A^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}\right] B^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} + A^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} \left[\nabla_e B^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}\right] \tag{2}$$

3. Commutativity with contraction

$$\nabla_d (A^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}) = \nabla_d A^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} \tag{3}$$

4. Consistency with the notion of tangent vector as directional derivatives on scalar fields. For $t \in V_p$ where V_p is the tangent space at a point p of a manifold M and $f \in \mathcal{T}$ where $\mathcal{T}$ is the set of smooth function from manifold M into $\mathbb{R}$.

$$t(f) = t^a \nabla_a f \tag{4}$$

5. Torsion free[1]

$$\nabla_a \nabla_b f = \nabla_b \nabla_a f \tag{5}$$

[1]This condition is not necessary, in the case it is not imposed, $\nabla_a \nabla_b f - \nabla_b \nabla_a f = -T^c{}_{ab} \nabla_c f$ where $T^c{}_{ab}$ is antisymmetric in a and b and is called the *torsion tensor*.

Any "rule" ∇, for producing new vector fields from old and which satisfy the five conditions above, is called by differential geometers a "*symmetric covariant derivative*". There are as many ways of defining a covariant derivative ∇ as there are of rearranging sources of the gravitational field. Different geodesics (free-fall trajectories) result from different distributions of masses and thus different definitions of ∇. This raises the question of uniqueness. From property (4) we know that any two derivative operators must agree in their action on scalar fields. What about the next lowest ranking tensor field? By comparing the difference of two derivative operators ∇ and $\tilde{\nabla}$ when acting on a dual vector field ω_b we find that the difference $(\nabla - \tilde{\nabla})$ defines a map of dual vectors at a point p to tensors of rank $\binom{1}{2}$ at p. Thus, given any any two derivative operators there exists a tensor field $C^c{}_{ab}$ such that

$$\nabla_a \omega_b = \tilde{\nabla}_a \omega_b - C^c{}_{ab} \omega_c \tag{6}$$

Where $C^c{}_{ab} = C^c{}_{ba}$. This displays the potential disagreement of ∇ and $\tilde{\nabla}$ on dual vector fields. For arbitrary tensor fields we find

$$\begin{aligned} \nabla_a T^{b_1 \cdots b_k}{}_{c_1 \cdots c_l} = \tilde{\nabla}_a T^{b_1 \cdots b_k}{}_{c_1 \cdots c_l} + \sum_i C^{b_i}{}_{ad} T^{b_1 \cdots d \cdots b_k}{}_{c_1 \cdots c_l} \\ - \sum_j C^d{}_{ac_j} T^{b_1 \cdots b_l}{}_{c_1 \cdots d \cdots c_l} \end{aligned} \tag{7}$$

Thus, the difference between the two derivative operators ∇ and $\tilde{\nabla}$ is completely characterized by the tensor field $C^c{}_{ab}$. We see that, given only the manifold structure, there are many distinct choices of derivative operator ∇, none preferred over the other. However, if we are given a metric g_{ab} on the manifold, a natural choice of derivative operator is uniquely picked out. With a metric in hand, it a very natural condition for parallel transport seemingly pops out. Consider two vectors v^a and u^a. If we parallel-transport these two vectors we would demand that their inner product remain invariant. After all, parallel transport should preserve angles and magnitudes between and of these vectors. If we consider a parallel transport in some direction $\mathbf{w}$ in the tangent plane

$$\nabla_{\mathbf{w}}(g_{ab} v^a u^b) = (\nabla_{\mathbf{w}} g_{ab}) v^a u^b + g_{ab}(\nabla_{\mathbf{w}} v^a) u^b + g_{ab} v^a (\nabla_{\mathbf{w}} u^b) = 0 \tag{8}$$

During parallel transport the components of v and u will remain the same ($\nabla_{\mathbf{w}} v^a = \nabla_{\mathbf{w}} u^b = 0$).

$$v^a u^b \nabla_{\mathbf{w}} g_{ab} = 0 \tag{9}$$

Thus, the dot product of two vector which are parallel transported will be preserved if and only if

$$\nabla_{\mathbf{w}} g_{ab} = 0 \tag{10}$$

This condition uniquely determines ∇. We can prove this by again considering two derivative operators ∇ and $\tilde{\nabla}$. From Eqs. (7) and (10) we have

$$0 = \nabla_a g_{bc} = \tilde{\nabla}_a g_{bc} - C^d{}_{ab} g_{dc} - C^d{}_{ac} g_{bd} \tag{11}$$

$$\tilde{\nabla}_a g_{bc} = C^d{}_{ab} g_{dc} + C^d{}_{ac} g_{bd} \tag{12}$$

$$\tilde{\nabla}_a g_{bc} = C_{cab} + C_{bac} \tag{13}$$

By renaming indices we also have

$$\tilde{\nabla}_b g_{ac} = C_{cba} + C_{abc} \tag{14}$$

$$\tilde{\nabla}_c g_{ba} = C_{acb} + C_{bca} \tag{15}$$

Adding the first two relations

$$C_{cab} + C_{bac} + C_{cba} + C_{abc} = \tilde{\nabla}_a g_{bc} + \tilde{\nabla}_b g_{ac} \tag{16}$$

$$2C_{cab} + C_{bac} + C_{abc} = \tilde{\nabla}_a g_{bc} + \tilde{\nabla}_b g_{ac} \tag{17}$$

Subtracting Eq.(15)

$$2C_{cab} + C_{bac} + C_{abc} - C_{acb} - C_{bca} = \tilde{\nabla}_a g_{bc} + \tilde{\nabla}_b g_{ac} - \tilde{\nabla}_c g_{ba} \tag{18}$$

$$2C_{cab} = \tilde{\nabla}_a g_{bc} + \tilde{\nabla}_b g_{ac} - \tilde{\nabla}_c g_{ba} \tag{19}$$

$$C^c{}_{ab} = \frac{1}{2} g^{cd} \left(\tilde{\nabla}_a g_{bd} + \tilde{\nabla}_b g_{ad} - \tilde{\nabla}_d g_{ba} \right) \tag{20}$$

This choice of $C^c{}_{ab}$ satisfies Eq.(10) uniquely. A metric g_{ab} then, naturally defines a derivative operator ∇.

As I said at the beginning of this section, we can use the path dependence of parallel transport to define an intrinsic notion of curvature. We do a similar calculation as above but for the action of two derivative operators. We find that the difference of interchanged orderings of the derivative operators can be expressed as a rank $\binom{1}{3}$ tensor field.

$$\nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c = R_{abc}{}^d \omega_d \tag{21}$$

$R_{abc}{}^d$ is called the *Riemann curvature tensor* and is directly related to the failure of a vector to return to its initial value when parallel transported around a small closed curve. For an arbitrary tensor field

$$\begin{aligned}(\nabla_a \nabla_b - \nabla_b \nabla_a) T^{c_1 \cdots c_k}{}_{d_1 \cdots d_l} = -\sum_{i=1}^{k} R_{abe}{}^{c_i} T^{c_1 \cdots e \cdots c_k}{}_{d_1 \cdots d_l} \\ \sum_{j=1}^{l} R_{abd_j}{}^{e} T^{c_1 \cdots c_k}{}_{d_1 \cdots e \cdots d_l}\end{aligned} \tag{22}$$

The Riemann tensor has the following properties

1. ${R_{abc}}^d = -{R_{bac}}^d$
2. ${R_{[abc]}}^d = 0$
3. For the derivative operator ∇_a naturally associated with the metric, $\nabla_a g_{bc} = 0$, we have
$$R_{abcd} = -R_{abdc} \tag{23}$$
4. The Bianchi identity holds[2]:
$$\nabla_{[a}{R_{bc]d}}^e = 0 \tag{26}$$
$$\frac{1}{6}(\nabla_a {R_{bcd}}^e - \nabla_b {R_{acd}}^e + \nabla_b {R_{cad}}^e + \nabla_c {R_{abd}}^e - \nabla_c {R_{bad}}^e - \nabla_a {R_{cbd}}^e) = 0 \tag{27}$$

It is useful to decompose the Riemann tensor into a 'trace part' and a 'trace free part'. From properties (1) and (3) the trace over its first and last two indices vanishes. However, the trace over the second and fourth (or first and third) define the *Ricci tensor*, R_{ac}
$$R_{ac} = {R_{abc}}^b \tag{28}$$
The Ricci tensor is a symmetric tensor $R_{ac} = R_{ca}$. The *scalar curvature*, R, is the trace over the Ricci tensor
$$R = {R_a}^a \tag{29}$$
The trace-free part of the Riemann tensor is called the *Weyl tensor* (conformal tensor), C_{abcd}. For manifold of dimensions $n \geq 3$ the Weyl tensor is defined by
$$R_{abcd} = C_{abcd} + \frac{2}{n-2}(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) - \frac{2}{(n-1)(n-2)} R g_{a[c}g_{d]b} \tag{30}$$

[2]Note the notation
$$T_{[a_1 \cdots a_l]} = \frac{1}{p!}\sum_\pi \delta_\pi T_{a_{\pi(1)} \cdots a_{\pi(p)}} \tag{24}$$
where the sum is taken over all permutations, π, of $1, \ldots, p$ and δ_π is $+1$ for even permutations and -1 for odd permutations. Brackets $[\cdots]$ indicate antisymmeterization over the indicies whereas parentheses $(\cdots)$ indicate symmeterization. For example,
$$T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba}) \qquad T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba}) \tag{25}$$
The $1/p!$ factor ensures that the symmeterization is of strength one i.e. if we symmetrize or antisymmetrize twice, the tensor remains the same $T_{((a_1 \cdots a_n))} = T_{(a_1 \cdots a_n)}$, $T_{[[a_1 \cdots a_n]]} = T_{[a_1 \cdots a_n]}$. The symmetric and antisymmetric tensors resulting from the tensor T are termed *cotensors* of T.

Contracting Eq.(27) gives an important relation. First contracting a and e

$$\frac{1}{6}(\nabla_a R_{bcd}{}^a - \nabla_b R_{acd}{}^a + \nabla_b R_{cad}{}^a + \nabla_c R_{abd}{}^a - \nabla_c R_{bad}{}^a - \nabla_a R_{cbd}{}^a) = 0 \tag{31}$$

$$\frac{1}{6}(\nabla_a R_{bcd}{}^a + \nabla_b R_{cad}{}^a + \nabla_b R_{cad}{}^a - \nabla_c R_{bad}{}^a - \nabla_c R_{bad}{}^a + \nabla_a R_{bcd}{}^a) = 0 \tag{32}$$

$$\nabla_a R_{bcd}{}^a + \nabla_b R_{cd} - \nabla_c R_{bd} = 0 \tag{33}$$

Raising the d index and contracting b and d gives

$$\nabla_a R_c{}^a + \nabla_b R_c{}^b - \nabla_c R = 0 \tag{34}$$

Or,

$$\nabla^a G_{ab} = 0 \tag{35}$$

where

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} \tag{36}$$

G_{ab} is known as the *Einstein tensor.*

Okay, so where does this lead us? Well, we now have the tools to compare arbitrarily ranked tensor fields in different coordinate frames or on manifolds with perturbed metrics. Let's now express the Riemann curvature tensor in terms of arbitrary connection coefficients. We start from Eq.(6) where we relate two differential operators acting on dual vector fields

$$\nabla_b \omega_c = \tilde{\nabla}_b \omega_c - C^d{}_{bc} \omega_d \tag{37}$$

Using Eq.(7)

$$\begin{aligned}
\nabla_a \nabla_b \omega_c = {} & \tilde{\nabla}_a (\tilde{\nabla}_b \omega_c - C^d{}_{bc} \omega_d) \\
& - C^d{}_{ab} \tilde{\nabla}_d \omega_c - C^d{}_{ac} \tilde{\nabla}_b \omega_d \\
& - C^d{}_{ab} C^b{}_{ce} \omega_d + C^d{}_{ab} C^c{}_{de} \omega_c \\
& + C^d{}_{ac} C^e{}_{bd} \omega_e + C^d{}_{ab} C^b{}_{ce} \omega_d
\end{aligned} \tag{38}$$

The first term on the third line cancels with the final term leaving us with

$$\begin{aligned}
\nabla_a \nabla_b \omega_c = {} & \tilde{\nabla}_a (\tilde{\nabla}_b \omega_c - C^d{}_{bc} \omega_d) \\
& - C^d{}_{ab} (\tilde{\nabla}_d \omega_c - C^c{}_{de} \omega_c) \\
& - C^d{}_{ac} (\tilde{\nabla}_b \omega_d - C^e{}_{bd} \omega_e)
\end{aligned} \tag{39}$$

Taking the difference with $\nabla_b \nabla_a$ and using the symmetry properties of $C^a{}_{bc}$ (we see immediately the second line will cancel because it is symmetric in ab)

$$\begin{aligned}(\nabla_a \nabla_b - \nabla_b \nabla_a)\omega_c = (\tilde{\nabla}_a \tilde{\nabla}_b - \tilde{\nabla}_b \tilde{\nabla}_a)\omega_c - (\tilde{\nabla}_a C^d{}_{bc} - \tilde{\nabla}_b C^d{}_{ac})\omega_d \\ - C^d{}_{ac}(\tilde{\nabla}_b \omega_d - C^e{}_{bd}\omega_e) \\ + C^d{}_{bc}(\tilde{\nabla}_a \omega_d - C^e{}_{ad}\omega_e)\end{aligned} \tag{40}$$

Of course we can rewrite $(\tilde{\nabla}_a \tilde{\nabla}_b - \tilde{\nabla}_b \tilde{\nabla}_a)\omega_c = \tilde{R}^d_{abc}\omega_d$. In the last line we swap $bc \to cb$ and rename $b \to a$

$$\begin{aligned}R_{abc}{}^d \omega_d = \tilde{R}^d_{abc}\omega_d - (\tilde{\nabla}_a C^d{}_{bc} - \tilde{\nabla}_b C^d{}_{ac})\omega_d \\ + (C^d{}_{ca}\tilde{\nabla}_b - C^d{}_{ac}\tilde{\nabla}_b)\omega_d \\ + (C^e{}_{ac}C^d{}_{be} - C^e{}_{bc}C^d{}_{ae})\omega_d\end{aligned} \tag{41}$$

Because this holds for any ω_d we can drop this factor. The second line cancels ($C^d{}_{ca} = C^d{}_{ac}$) and we are left with the following relation

$$R_{abc}{}^d = \tilde{R}^d_{abc} + 2\left[-\tilde{\nabla}_{[a}C^d{}_{b]c} + C^e{}_{c[a}C^d{}_{b]e}\right] \tag{42}$$

1.1 Conformal transformations

Now we look to put it all together by comparing the quantities we have computed above under conformal transformations. Let M be an n-dimensional manifold with metric g_{ab} of any signature. If W is a smooth, strictly positive function, then the metric $\tilde{g}_{ab} = W^2 g_{ab}$ is said to arise from g_{ab} via a *conformal transformation*. Both metrics obey the same causal structures. Their inverses are given by g^{ab} and $\tilde{g}^{ab}$ and $\tilde{g}^{ab} = W^{-2}g^{ab}$ such that $\tilde{g}^{ab}\tilde{g}_{bc} = W^2 W^{-2} g^{ab} g_{bc} = \delta^a{}_c$. Each metric is accompanied by a derivative operator ∇ with g_{ab} and $\tilde{\nabla}$ with $\tilde{g}_{ab}$ such that $\nabla_a g_{bc} = \tilde{\nabla}_a \tilde{g}_{bc} = 0$. The two operators are related via Eq.(7). Interchanging ∇ and $\tilde{\nabla}$ i.e. ($\tilde{\nabla}_a \omega_b = \nabla_a \omega_b - C^c{}_{ab}\omega_c$) we see that the connection coefficients $C^c{}_{ab}$ are now given by Eq.(20)

$$C^c{}_{ab} = \frac{1}{2}\tilde{g}^{cd}\left(\nabla_a \tilde{g}_{bd} + \nabla_b \tilde{g}_{ad} - \nabla_d \tilde{g}_{ba}\right) \tag{43}$$

$$= \frac{1}{2}W^{-2}g^{cd}\left(\nabla_a(W^2 g_{bd}) + \nabla_b(W^2 g_{ad}) - \nabla_d(W^2 g_{ba})\right) \tag{44}$$

Noting $\nabla_a g_{bc} = 0$

$$= W^{-1}g^{cd}\left(g_{bd}\nabla_a W + g_{ad}\nabla_b W - g_{ba}\nabla_d W\right) \tag{45}$$

$$= g^{cd}\left(g_{bd}\nabla_a \ln W + g_{ad}\nabla_b \ln W - g_{ba}\nabla_d \ln W\right) \tag{46}$$

$$= \delta^c{}_b\nabla_a \ln W + \delta^c{}_a\nabla_b \ln W - g^{cd}g_{ba}\nabla_d \ln W \tag{47}$$

$$= 2\delta^c{}_{(b}\nabla_{a)} \ln W - g^{cd}g_{ba}\nabla_d \ln W \tag{48}$$

Of particular interest is the relation of curvature $\tilde{R}^d_{abc}$ associated with $\tilde{\nabla}$ and $R_{abc}{}^d$ associated with ∇. We have already done this computation! The result is given in Eq.(42)

$$\tilde{R}^d_{abc} = R_{abc}{}^d + 2\left[-\nabla_{[a}C^d{}_{b]c} + C^e{}_{c[a}C^d{}_{b]e}\right] \tag{49}$$

$$= R_{abc}{}^d - \nabla_a C^d{}_{bc} + \nabla_b C^d{}_{ac} + C^e{}_{ac}C^d{}_{be} - C^e{}_{bc}C^d{}_{ae} \tag{50}$$

$$= R_{abc}{}^d + (C^e{}_{ac}C^d{}_{be} - \nabla_a C^d{}_{bc}) - (C^e{}_{bc}C^d{}_{ae} - \nabla_b C^d{}_{ac}) \tag{51}$$

Plugging in our derived connection coefficients from Eq.(48) explicitly term by term

$$\nabla_a C^d{}_{bc} = \nabla_a\left[(\delta^d{}_c\nabla_b + \delta^d{}_b\nabla_c - g^{de}g_{cb}\nabla_e)\ln W\right] \tag{52}$$

$$= \delta^d{}_c\nabla_a(\nabla_b \ln W) + \delta^d{}_b\nabla_a(\nabla_c \ln W) - g^{de}g_{cb}\nabla_a(\nabla_e \ln W) \tag{53}$$

$$C^e{}_{ac}C^d{}_{be} = (\delta^e{}_c\nabla_a \ln W + \delta^e{}_a\nabla_c \ln W - g^{ef}g_{ca}\nabla_f \ln W)(\delta^d{}_e\nabla_b \ln W + \delta^d{}_b\nabla_e \ln W - g^{dh}g_{eb}\nabla_h \ln W) \tag{54}$$

$$\begin{aligned}&= \delta^d{}_c(\nabla_a \ln W)\nabla_b \ln W + \delta^d{}_b(\nabla_a \ln W)\nabla_c \ln W - (\nabla_a \ln W)g_{bc}g^{df}\nabla_f \ln W\\ &+ \delta^d{}_a(\nabla_c \ln W)\nabla_b \ln W + \delta^d{}_b(\nabla_c \ln W)\nabla_a \ln W - (\nabla_c \ln W)g_{ab}g^{dh}\nabla_h \ln W\\ &- g_{ca}g^{df}(\nabla_f \ln W)\nabla_b \ln W - g_{ca}\delta^d{}_b g^{ef}(\nabla_f \ln W)\nabla_e \ln W + (\nabla_b \ln W)g_{ca}g^{df}\nabla_f \ln W\end{aligned} \tag{55}$$

$$\nabla_b C^d{}_{ac} = \text{Eq.(52) with } a \Leftrightarrow b \tag{56}$$

$$C^e{}_{bc}C^d{}_{ae} = \text{Eq.(55) with } a \Leftrightarrow b \tag{57}$$

Putting everything together and noting from Eq.(51) that the full expression is asymmetric in a and b

$$\begin{aligned}\tilde{R}^d_{abc} = R_{abc}{}^d + 2\Big[&\delta^d{}_{[a}\nabla_{b]}\nabla_c \ln W - g^{de}g_{c[a}\nabla_{b]}\nabla_e \ln W\\ &+ (\nabla_{[a} \ln W)\delta^d{}_{b]}\nabla_c \ln W - (\nabla_{[a} \ln W)g_{b]c}g^{df}\nabla_f \ln W\\ &- g_{c[a}\delta^d{}_{b]}g^{ef}(\nabla_e \ln W)\nabla_f \ln W\Big]\end{aligned} \tag{58}$$

Tracing over b and d gives the Ricci tensor

$$\begin{aligned}\tilde{R}_{ac} = R_{ac} - (n-2)\nabla_a \nabla_c \ln W - g_{ac} g^{de} \nabla_d \nabla_e \ln W \\ + (n-2)(\nabla_a \ln W)\nabla_c \ln W - (n-2) g_{ac} g^{de} (\nabla_d) \ln W \nabla_e \ln W\end{aligned} \tag{59}$$

Contracting the above equation with $\tilde{g}^{ac} = W^{-2} g^{ac}$ give the scalar curvature

$$\tilde{R} = W^{-2}\left[R - 2(n-1) g^{ac} \nabla_a \nabla_c \ln W - (n-2)(n-1) g^{ac} (\nabla_a \ln W) \nabla_c \ln W\right] \tag{60}$$

Eqs.(58)(59)(60) describe how curvature is changed by conformal transformations.

1.2 Computing Curvature with a coordinate frame

Upon choosing a coordinate frame we can write

$$R_{\mu\nu\rho}{}^{\sigma} = \frac{\partial}{\partial x^{\nu}} \Gamma^{\sigma}{}_{\nu\rho} - \frac{\partial}{\partial x^{\mu}} \Gamma^{\sigma}{}_{\nu\rho} + \Gamma^{\alpha}{}_{\mu\rho} \Gamma^{\sigma}{}_{\alpha\nu} - \Gamma^{\alpha}{}_{\nu\rho} \Gamma^{\sigma}{}_{\alpha\mu} \tag{61}$$

$$R_{\mu\rho} = R_{\mu\nu\rho}{}^{\nu} = \frac{\partial}{\partial x^{\nu}} \Gamma^{\nu}{}_{\mu\rho} - \frac{\partial}{\partial x^{\mu}} \Gamma^{\nu}{}_{\nu\rho} + \Gamma^{\alpha}{}_{\mu\rho} \Gamma^{\nu}{}_{\alpha\nu} - \Gamma^{\alpha}{}_{\nu\rho} \Gamma^{\nu}{}_{\alpha\mu} \tag{62}$$

$$R = R_{\mu}{}^{\mu} = \frac{\partial}{\partial x^{\nu}} \Gamma^{\nu\mu}{}_{\mu} - \frac{\partial}{\partial x^{\mu}} \Gamma^{\nu\mu}{}_{\nu} + \Gamma^{\alpha\mu}{}_{\mu} \Gamma^{\nu}{}_{\nu\alpha} - \Gamma^{\alpha\mu}{}_{\nu} \Gamma^{\nu}{}_{\alpha\mu} \tag{63}$$

In the coordinate basis the components of the Christoffel symbol are given by

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left[\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right] \tag{64}$$

We note that,

$$\Gamma^{\nu}{}_{\nu\mu} = \frac{1}{2} g^{\mu\alpha} \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} \tag{65}$$

Using the formula for the inverse of a matrix,

$$g^{\mu\alpha} \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} = \frac{1}{g} \frac{\partial g}{\partial x^{\mu}} \tag{66}$$

And thus,

$$\Gamma^{\nu}{}_{\nu\mu} = \frac{1}{2} \frac{1}{g} \frac{\partial g}{\partial x^{\mu}} \tag{67}$$

2 Randall-Sundrum Model

The scale of electroweak symmetry breaking: 246 GeV. The Planck scale: 10^{19} GeV. RS models attempt to resolve this "hierarchy" of scales. RSI models begin by adding a fifth spatial dimension which is compactified onto a $S^1/\mathbb{Z}_2$ orbifold (circle with two hemispheres identified with each other $\theta = -\theta$). This results in a mandatory assignment of parity charges for all fields as well as an invariance of the action under this parity transformation (due to the arbitrary assignment of parity charge). Even with the fifth dimension curved we still suspect that at every point in the fifth dimension the other four dimensions will obey local Poincare invariance. Thus, all along the compactified fifth dimension the induced metric must still be the familiar flat Minkowski metric $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. The 5-D metric will be a function of the fifth coordinate y. The most general ansatz for the five-dimensional metric is given by

$$ds^2 = \Omega(y)dx^\mu dx^\nu \eta_{\mu\nu} - dy^2 \tag{68}$$

The amount of curvature is directly related to the function $\Omega(y)$ and thus is called the warp factor. From the discussion above, in the absence of the fifth dimension, the metric must obey the usual Minkowski metric giving the condition $\Omega(0) = 1$. Our job now is to calculate $\Omega(y)$. We begin by transforming to a conformally flat frame (a frame in which all metrics are related via overall rescalings)

$$dy = \Omega(z)^{1/2}dz \tag{69}$$

Giving the metric

$$\begin{aligned} ds^2 &= \Omega(z)(dx^\mu dx^\nu \eta_{\mu\nu} - dz^2) \\ &= \Omega(z)\eta_{MN}dx^M dx^N \end{aligned} \tag{70}$$

Where η_{MN} is the flat 5-D metric $\eta_{MN} = \text{diag}(1, -1, -1, -1, -1)$. This metric is invariant under rescaling of the coordinates ($x^\mu \to \alpha x^\mu$, $z \to \alpha z$) assuming $\Omega(z) \to \Omega(\alpha z) = \Omega(z)/\alpha^2$. From our previous sections we see that $\Omega(z)$ just a conformal transformation i.e. $g_{MN} = \Omega(z)\tilde{g}_{MN}$. This opens all of the tools in the previous section. Eqs.(59)(60) allow us to relate the Einstein tensor $G_{MN} = R_{MN} - \frac{1}{2}g_{MN}R$ under the two different metrics g_{MN} and $\tilde{g}_{MN}$ in any number of dimensions n. We redefine our conformal transformation

$$\Omega(z) \to e^{-\omega(z)} \tag{71}$$

for convenience and with the expressions in Eqs.(59)(60) in mind. The relation between the two Einstein tensors reads

$$G_{MN} = \tilde{G}_{MN} + \frac{d-2}{2}\left[\frac{1}{2}\tilde{\nabla}_M\omega\tilde{\nabla}_N\omega + \tilde{\nabla}_M\tilde{\nabla}_N\omega - \tilde{g}_{MN}\left(\tilde{\nabla}_K\tilde{\nabla}^K\omega - \frac{d-3}{4}\tilde{\nabla}_K\omega\tilde{\nabla}^K\omega\right)\right] \tag{72}$$

Where the covariant derivative $\tilde{\nabla}$ are with respect to the metric $\tilde{g}$. The Einstein tensor on the left hand side is computed with the g metric. In our calculation, $d = 5$, $\tilde{g}_{MN} = \eta_{MN}$ and thus, $\tilde{\nabla}_M \to \partial_M$.

$$G_{MN} = \tilde{G}_{MN} + \frac{3}{2}\left[\frac{1}{2}\partial_M\omega\partial_N\omega + \partial_M\partial_N\omega - \eta_{MN}\left(\partial_K\partial^K\omega - \frac{1}{2}\partial_K\omega\partial^K\omega\right)\right] \tag{73}$$

The only non-zero terms coming from derivatives are those acting on the fifth dimension $\partial_\mu\omega(z) = 0$. Thus, we are left with components G_{55} and $G_{\mu\nu}$

$$G_{55} = \tilde{G}_{55} + \frac{3}{2}\left[\frac{1}{2}(\partial_5\omega)^2 + \partial_5^2\omega - \partial_5^2\omega + \frac{1}{2}(\partial_5\omega)^2\right] \tag{74}$$

$$= \tilde{G}_{55} + \frac{3}{2}(\partial_5\omega)^2 \tag{75}$$

$$G_{\mu\nu} = \tilde{G}_{\mu\nu} - \frac{3}{2}\eta_{\mu\nu}\left(-\partial_5^2\omega + \frac{1}{2}(\partial_5\omega)^2\right) \tag{76}$$

In the vacuum (flat metric) the Einstein tensor reduces to zero, $\tilde{G}_{MN} = 0$. Defining $\partial_5\omega \equiv \omega'$ we are left with the following Einstein tensor components

$$G_{55} = \frac{3}{2}\omega'^2 \tag{77}$$

$$G_{\mu\nu} = -\frac{3}{2}\eta_{\mu\nu}(-\omega'' + \frac{1}{2}\omega'^2) \tag{78}$$

Einstein's field equation is written as

$$G_{MN} = \kappa^2 T_{MN} \tag{79}$$

Where κ is the Einstein gravitational constant. In $5D$ the gravitational constant is related to the $5D$ Planck scale by

$$\kappa^2 = \frac{1}{M_*^3} \tag{80}$$

The five dimensional action for gravity with a bulk cosmological constant is given by

$$S = -\int d^5x\sqrt{g}(M_*^3 R + \Lambda) \tag{81}$$

The variation of the action with respect to the scalar curvature results in Einstein's tensor. For the second term we have

$$S_\Lambda = -\int d^5x\sqrt{g}\Lambda \tag{82}$$

Generically the stress-energy tensor is defined as the variation of the Lagrangian with respect to the metric tensor

$$T_{MN} = \frac{\delta L}{\delta g^{MN}} \tag{83}$$

Given Eq.(81) and the useful identity for some non-singular matrix A

$$\delta(\det A) = \det A \mathrm{Tr}\left[\delta(A)A^{-1}\right] \tag{84}$$

Yielding the result

$$\delta(\sqrt{\det g_{MN}}) = \frac{1}{2\sqrt{\det g_{MN}}}\delta(\det g_{MN}) = \frac{\det g_{MN}}{2\sqrt{\det g_{MN}}}\mathrm{Tr}\left[\delta(g_{MN})g^{MN}\right] \tag{85}$$

Because the metric is diagonal we are left with ($\det g_{MN} \equiv g$)

$$\delta(\sqrt{g}) = \frac{1}{2}\sqrt{g}g^{MN}\delta g_{MN} = -\frac{1}{2}\sqrt{g}g_{MN}\delta g^{MN} \tag{86}$$

Taking the variation of the Lagrangian above

$$\delta L = -\delta(\sqrt{g})\Lambda = \frac{1}{2}\sqrt{g}g_{MN}\delta g^{MN}\Lambda \tag{87}$$

Noting[3],

$$\det g_{MN} = \prod_{i=0}^{5} g_{ii} = 1, \qquad i = 0,1,2,3,5 \tag{88}$$

Thus, the variation of the Lagrangian with respect to the metric, i.e. the stress-energy tensor, is given by

$$T_{MN} = \frac{\delta L}{\delta g^{MN}} = \frac{1}{2}\Lambda g_{MN} \tag{89}$$

[3]Also note that in the case of the flat Minkowski metric $g = \det g_{\mu\nu} = -1$ in which case our volume element factor would be defined as $\sqrt{-g}$

This stress-energy can be interpreted as the back-reaction of gravity due to the tension (energy density) of the branes. We now have Einstein's field equations

$$G_{MN} = \frac{1}{2M_*^3}\Lambda g_{MN} \tag{90}$$

We see that the 5-5 component gives us the relation

$$\frac{3}{2}\omega'^2 = \frac{\Lambda}{2M_*^3} g_{55} \tag{91}$$

$$\frac{3}{2}\omega'^2 = \frac{\Lambda}{2M_*^3} e^{-\omega(z)}\eta_{55} \tag{92}$$

$$\frac{3}{2}\omega'^2 = -\frac{\Lambda}{2M_*^3} e^{-\omega(z)} \tag{93}$$

$$\omega' e^{\omega(z)/2} = \sqrt{-\frac{\Lambda}{3M_*^3}} \tag{94}$$

This yields the solution

$$\omega(z) = 2\ln\left[C + z\sqrt{-\frac{\Lambda}{12M_*^3}}\right] \tag{95}$$

$$e^{\omega(z)/2} = C + z\sqrt{-\frac{\Lambda}{12M_*^3}} \tag{96}$$

where C is a constant which we fix using the required conformal symmetry constraint

$$e^{-\omega(\alpha z)} = \frac{1}{\alpha^2} e^{-\omega(z)} \tag{97}$$

implying $C = 0$ or if $C \neq 0$ under conformal transformations it is rescaled as $C \to C/\alpha^2$. We note that the argument of the natural logarithm must be positive, and thus we require $\Lambda < 0$, $z > 0$. The first requirement indicates that we are working in an anti-de Sitter space, and the latter is enforced via our compactification scheme on the $S^1/\mathbb{Z}_2$ orbifold, requiring $z = -z$. Defining the positive constant $k^2 \equiv -\Lambda/12M_*^3$ we have the following solution

$$e^{-\omega(z)} = \frac{1}{k^2|z|^2} \tag{98}$$

We should note that the transformation diverges as $z \to 0$ and thus we will need a regularization procedure to define the transformation on the boundary.

We have found a metric which is a solution to the radial component of the Einstein field equation. Now we need to check that the metric solution is fully consistent with bulk Einstein field equations. The bulk field equations are given in Eq.(78) and we can equate these with the bulk cosmological constant energy density derived in Eq.(90)

$$\frac{3}{2}\eta_{\mu\nu}(\omega'' - \frac{1}{2}\omega'^2) = \frac{1}{2M_*^3}\Lambda g_{\mu\nu} \tag{99}$$

$$\frac{3}{2}\eta_{\mu\nu}(\omega'' - \frac{1}{2}\omega'^2) = \frac{1}{2M_*^3}\Lambda e^{\omega(z)}\eta_{\mu\nu} \tag{100}$$

$$\frac{3}{2}(\omega'' - \frac{1}{2}\omega'^2) = \frac{\Lambda}{2M_*^3}\frac{1}{k^2|z|^2} \tag{101}$$

$$\frac{3}{2}(\omega'' - \frac{1}{2}\omega'^2) = -\frac{6}{|z|^2}. \tag{102}$$

Our solution, because it contains an absolute value, contains a delta-function contribution within it's second derivative [TM: Re-frame this in the language of junction conditions]

$$\omega'' = -\frac{2}{z^2} + \frac{4}{|z|}(\delta(z) - \delta(z - z_{\mathrm{IR}})) \tag{103}$$

giving the bulk field equation

$$-\frac{6}{|z|^2} + \frac{6}{|z|}\big(\delta(z) - \delta(z - z_{\mathrm{IR}})\big) \neq -\frac{6}{|z|^2}. \tag{104}$$

This indicates that in order for Eq.(98) to be a valid solution, we require each brane to carry additional energy density to cancel these delta-function contributions.

2.1 Radius Stabilization

In the above section, the magnitude or radius of the extra dimension was ad hoc, defined in such a way to solve the hierarchy problem. We should note, however, in the absence of a potential along the radial dimension, we are left with a moduli space consisting of a single flat direction that may generate a massless scalar field known as the *radion* from excitations perpendicular to the radial dimension. A massless scalar particle is phenomenologically unviable and hence we require a mechanism to give the radion a mass and dynamically stabilize the size of the extra dimension. The simplest dynamical solution for radius stabilization was proposed by Goldberger and Wise (GW). The proposal was this: introduce a bulk scalar field in the RS setup with a mass generated by means of a non-trivial potential along the radial coordinate between the two branes.

To realize GW mechanism we denote the scalar field as Φ and consider the following action

$$S=\int d^5x\sqrt{g}\left(-M_*^3R+\frac{1}{2}\nabla\Phi\nabla\Phi-V(\Phi)\right)-\int d^4x\sqrt{g_{\rm UV}}\lambda_{\rm UV}(\Phi)-\int d^4x\sqrt{g_{\rm IR}}\lambda_{\rm IR}(\Phi) \tag{105}$$

where the first term is the usual Einstein-Hilbert action and the final two terms denote the brane induced potentials for the scalar field on the UV and IR branes. Just as in the RS set-up we consider the looks for a metric solving Einstein's equations with the following ansatz

$$ds^2=e^{-2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu-dy^2 \tag{106}$$

Instead of deriving Einstein's equations as done in the previous sections we will first do the computation via the more 'traditional' methods for explicitness. Once we have a result, we will check these results against those obtained using the other method. By 'traditional' methods I mean explicitly computing the components of the Einstein tensor via the Riemann tensor. We have the following diagonal metric

$$g_{MN}=\begin{pmatrix} e^{-2A(r)} & & & & \\ & -e^{-2A(r)} & & & \\ & & -e^{-2A(r)} & & \\ & & & -e^{-2A(r)} & \\ & & & & -1 \end{pmatrix} \tag{107}$$

along with the following definitions

$$R^\mu{}_{\nu\alpha\beta}=\partial_\alpha\Gamma^\mu{}_{\nu\beta}-\partial_\beta\Gamma^\mu{}_{\nu\alpha}+\Gamma^\mu{}_{\rho\alpha}\Gamma^\rho{}_{\nu\beta}-\Gamma^\mu{}_{\rho\beta}\Gamma^\rho{}_{\nu\alpha} \tag{108}$$

$$R_{\mu\nu}=R^\alpha{}_{\mu\alpha\nu}=\partial_\alpha\Gamma^\alpha{}_{\mu\nu}-\partial_\nu\Gamma^\alpha{}_{\mu\alpha}+\Gamma^\alpha{}_{\rho\alpha}\Gamma^\rho{}_{\mu\nu}-\Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\mu\alpha} \tag{109}$$

$$\Gamma^\rho{}_{\mu\nu}=\frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma}+\partial_\nu g_{\mu\sigma}-\partial_\sigma g_{\mu\nu}\right) \tag{110}$$

$$G_{MN}=R_{MN}-\frac{1}{2}g_{MN}R. \tag{111}$$

Starting with the first term of Eq.(109):

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right) \tag{112}$$

We are interested in the equations of motion along the radial and bulk directions (the time equations of motion will give the same information as the bulk evolution) thus we check to see if we have non-zero components. The only non-zero contribution from the first term comes from the bulk component differentiated with respect to the radial direction ($M = i, N = i, \alpha = r$)

$$(-1)\partial_r \Gamma^{r}{}_{ii} = -\frac{1}{2} \partial_r \left\{ g^{r\sigma} (\partial_i g_{i\sigma} + \partial_i g_{\sigma} - \partial_\sigma g_{ii}) \right\} \tag{113}$$

when $\sigma = r$

$$= -\frac{1}{2} \partial_r \left\{ -g^{rr} \partial_r g_{ii} \right\} = \partial_r \{ A' e^{-2A} \} = A'' e^{-2A} - 2A'^2 e^{-2A}. \tag{114}$$

The second term does not contribute

$$\partial_\nu \Gamma^{\alpha}{}_{\mu_\alpha} = 0 \tag{115}$$

Likewise, the third term does not contribute

$$\Gamma^{\alpha}{}_{\rho\alpha} \Gamma^{\rho}{}_{\mu\nu} = 0 \tag{116}$$

The final term contributes to both the bulk and radial equations of motion, expanded in full we have the following

$$\begin{aligned} \Gamma^{\alpha}{}_{\rho\nu} \Gamma^{\rho}{}_{\mu\alpha} = \frac{1}{4} g^{\alpha\sigma} g^{\rho\gamma} \Big[& \partial_\rho g_{\nu\sigma} \partial_\mu g_{\alpha\gamma} + \partial_\rho g_{\nu\sigma} \partial_\alpha g_{\mu\gamma} - \partial_\rho g_{\nu\sigma} \partial_\gamma g_{\mu\alpha} \\ & + \partial_\nu g_{\rho\sigma} \partial_\mu g_{\alpha\gamma} + \partial_\nu g_{\rho\sigma} \partial_\alpha g_{\mu\gamma} - \partial_\nu g_{\rho\sigma} \partial_\gamma g_{\mu\alpha} \\ & - \partial_\sigma g_{\rho\nu} \partial_\mu g_{\alpha\gamma} - \partial_\sigma g_{\rho\nu} \partial_\alpha g_{\mu\gamma} + \partial_\sigma g_{\rho\nu} \partial_\gamma g_{\mu\alpha} \Big] \end{aligned} \tag{117}$$

where the terms in red are those which contain non-zero contributions. Let's iterate through each term:

$$-\frac{1}{4} g^{\alpha\sigma} g^{\rho\gamma} \partial_\rho g_{\nu\sigma} \partial_\gamma g_{\mu\alpha} \tag{118}$$

this will give a non-zero contribution when $\alpha = i, \sigma = i, \rho = r, \gamma = r, \mu = i, \nu = i$

$$-\frac{1}{4} g^{ii} g^{rr} \partial_r g_{ii} \partial_r g_{ii} = -e^{-2A} A'^2. \tag{119}$$

The second non-zero term

$$\frac{1}{4}g^{\alpha\sigma}g^{\rho\gamma}\partial_\nu g_{\rho\sigma}\partial_\mu g_{\alpha\gamma} \tag{120}$$

will give a contribution when $\alpha = i, \sigma = i, \rho = i, \gamma = i, \mu = r, \nu = r$

$$\frac{1}{4}g^{ii}g^{ii}\partial_r g_{ii}\partial_r g_{ii} = A'^2 \tag{121}$$

The third non-zero term

$$-\frac{1}{4}g^{\alpha\sigma}g^{\rho\gamma}\partial_\sigma g_{\rho\nu}\partial_\alpha g_{\mu\gamma} \tag{122}$$

will give a contribution when $\alpha = r, \sigma = r, \rho = i, \gamma = i, \mu = i, \nu = i$

$$-\frac{1}{4}g^{rr}g^{ii}\partial_r g_{ii}\partial_r g_{ii} = -e^{-2A}A'^2 \tag{123}$$

Putting it all together we have the following Riemann tensor components:

$$R_{rr} = A'^2 \tag{124}$$

$$R_{ii} = A''e^{-2A} - 4A'^2e^{-2A} \tag{125}$$

The scalar curvature is given by

$$R = g^{MN}R_{MN} = e^{-2A}R_{tt} - 3(A'' - 4A'^2) - A'^2 \tag{126}$$

To be continued...

References

[1] Wald, Robert M. General relativity. University of Chicago press, 2010.

[2] Misner, Charles W., Kip S. Thorne, and John Archibald Wheeler. Gravitation. Macmillan, 1973.