MAGNETIC WAVES VIA ROTATING, ELECTRICALLY CONDUCTING FLUIDS

November 20, 2018

Tony Menzo

I provide a detailed derivation of wave equations for four different force balances on a electrically conducting fluid parcel within a rotating, infinite plane geometry immersed in a uniform magnetic field. The different force balances reveal four different wave motions. Namely; Inertial, Alfven, Magnetic-Coriolis(MC), and Magnetic-Archimedes-Coriolis(MAC) waves. I then implement a plane wave solution into these wave equations to find dispersion relations to further the understand the wave properties.

1 Introduction

Earth's magnetic field experiences temporal changes in its intensity, these changes range over timescales from milliseconds to millions of years (geomagnetic pole reversals). Diurnal variations refer to external-sourced temporal intensity changes on the order of milliseconds to days. Interactions between solar radiation and the Earth's magnetosphere and ionosphere create streams of current that produce their own magnetic field, these fields interact with the geomagnetic field and produces very sudden changes to the core field. Secular variation (SV) is described by intensity changes of the geomagnetic field on timescales of a year or more. General consensus is that these variations are of internal-origin, more specifically originating from variations in fluid flow within the outer core as well as at the core mantle boundary (CMB). These secular variations are generally accepted to be of internal origin and appear to have wave like properties(Alexandrescu et al 1995, 1996; Le Huy et al 1998; Bloxham et al 2002, Chambodut and Mandea 2005; De Michelis and Tozzi 2005). For example the phenomena known as the geomagnetic jerk is thought to be caused by these magnetic waves generated at the core mantle boundary(CMB)(Malkin 2013,2015,2016). Much theoretical work has been done on this subject over the past seventy years, starting with Hannes Alfven in 1942 when he predicted the existence of electromagnetic-hydromagnetic waves in plasmas, now known as Alfven waves. Since then many great geophysicists and applied mathematicians have succeeded in developing the mathematical framework necessary to describe wave motions possibly produced within the Earth's outer core (Braginsky (1964; 1967), Chandrasekhar (1961), Hide (1966), Malkus (1967), Moffatt (1978), Melchoir (1986), Finlay (2005), Davidson (2001)).

Much of the basic wave motion properties of a rotating, electrically conducting, incompressible fluid can be mathematically described by thinking about the force balance on a fluid parcel within the a rotating infinite plane layer. Coriolis, Lorentz, and buoyancy forces can be expected to be present. Because the fluid parcel has some mass and velocity relative to the rotating plane it possesses some inertia which will resist any force attempting to slow it down as described by Newton's first law. Having elucidated the relevant forces that a rotating, electrically conducting fluid parcel could experience in a non-diffusive, non-convecting fluid we can easily imagine four different force balances, all of which produce different wave motions with different wave properties. This will be the main focus of what is to follow. Before though, we should take a moment to appreciate the complexity of the motion experienced by this fluid parcel which can give us some insight into the very complex motions of the outer core that generate the Earth's magnetic field.

I will derive various wave equations emerging due to four different force balances. Then I assume a plane wave solution and find dispersion relations for each. I start with the simple case of inertial wave and Alfven waves and then continue to add various forces to find wave equations for MC and MAC waves. By manipulating the Navier-Stokes equation, magnetic induction equation and the heat equation I am able to produce these wave equations of the form

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = c^2 \nabla^2 \mathbf{u}$$

With these derivations I attempt to build a better intuition of the relevant wave motions present within the core and how they may be revealed at the surface expressed as secular variation.

2 Methods

I began my investigation by deriving four wave equations using the Navier-Stokes Equation for a rotating magnetohydrodynamic fluid, the magnetic induction equation and the Heat equation.

$$\rho_o \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2(\mathbf{\Omega} \times \mathbf{u}) \right] = -\nabla P - \rho_o \alpha_v T \mathbf{g} + \frac{1}{\mu_0}(\mathbf{J} \times \mathbf{B}) + \rho_o \nu \nabla^2 \mathbf{u}$$
$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$
$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla)T = \kappa \nabla^2 T + \varepsilon$$

We begin with the simplest case first, namely inertial waves. For all of the wave equations I assume an infinite plane geometry. Inertial waves are present when the Coriolis forces exceed the Lorentz force experienced by the fluid parcel. In this case the Navier-Stokes equation (1) reduces to

$$\frac{\partial \mathbf{u}}{\partial t} + 2(\mathbf{\Omega} \times \mathbf{u}) = -\frac{1}{\rho_o}\nabla P$$

We are only interested in leading order terms in $\mathbf{u}$ so the $(\mathbf{u} \bullet \nabla)\mathbf{u}$ term is neglected as well as the $\rho_o \nu \nabla^2 \mathbf{u}$ term. In order to achieve a wave equation from this we need to take the curl twice as well as a time derivative.

Taking one curl...

$$\frac{\partial(\nabla \times \mathbf{u})}{\partial t} + 2(\nabla \times (\mathbf{\Omega} \times \mathbf{u})) = -\frac{1}{\rho_o}(\nabla \times \nabla P)$$

Noticing the curl of the gradient = 0 the pressure term goes away and realizing in an incompressible fluid $(\boldsymbol{\nabla} \times (\mathbf{\Omega} \times \mathbf{u})) = -(\mathbf{\Omega} \cdot \nabla)\mathbf{u}$ we get

$$\frac{\partial(\nabla \times \mathbf{u})}{\partial t} - 2(\mathbf{\Omega} \cdot \nabla)\mathbf{u} = 0$$

Taking a time derivative yields

And another curl

$$\frac{\partial^2(\nabla \times \mathbf{u})}{\partial t^2} = 2(\mathbf{\Omega} \cdot \nabla)\frac{\partial \mathbf{u}}{\partial t}$$

$$\frac{\partial^2(\nabla \times \nabla \times \mathbf{u})}{\partial t^2} = 2(\mathbf{\Omega} \cdot \nabla)\frac{\partial(\nabla \times \mathbf{u})}{\partial t}$$

Seeing that $\boldsymbol{\nabla}\nabla\boldsymbol{\nabla}\mathbf{uu} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) - \nabla^2\mathbf{u} = -\boldsymbol{\nabla}^2\mathbf{u}$ and replacing the time derivative of $\boldsymbol{\nabla}\mathbf{xu}$ with what we found a few step earlier yields our wave equation for an inertial wave.

$$\frac{\partial^2 \left(\nabla^2 \mathbf{u}\right)}{\partial t^2} = -4(\mathbf{\Omega} \cdot \nabla)^2\mathbf{u}$$

Next we derive a wave equation for Alfven wave which are present when the magnetic forces dominate over the coriolis forces. In this case we linearise (1) and reduce it to

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_o}\nabla P + \frac{1}{\rho_o \mu}\left(\mathbf{B_o}\cdot\nabla\right)\mathbf{b}$$

We also will need the linearised induction equation

Taking the curl of both

$$\frac{\partial \mathbf{b}}{\partial t} = \left(\mathbf{B_o}\cdot\nabla\right)\mathbf{u}$$

$$\frac{\partial(\nabla\times\mathbf{u})}{\partial t} = -\frac{1}{\rho_o}(\nabla\times\nabla P) + \frac{1}{\rho_o \mu}\left(\mathbf{B_o}\cdot\nabla\right)(\nabla\times\mathbf{b})$$
$$\frac{\partial(\nabla\times\mathbf{b})}{\partial t} = \left(\mathbf{B_o}\cdot\nabla\right)(\nabla\times\mathbf{u})$$

Then taking a time derivative of both

$$\frac{\partial^2(\nabla\times\mathbf{u})}{\partial t^2} = \frac{1}{\rho_o \mu}\left(\mathbf{B_o}\cdot\nabla\right)\frac{\partial(\nabla\times\mathbf{b})}{\partial t}$$

And replacing the time derivative of the curl of the magnetic field with our above result give us a wave equation for an Alfven wave

$$\frac{\partial^2(\nabla\times\mathbf{u})}{\partial t^2} = \frac{1}{\rho_o \mu}\left(\mathbf{B_o}\cdot\nabla\right)^2(\nabla\times\mathbf{u})$$

Now that we have derived wave equations for situations where magnetic forces or coriolis forces dominate, we will consider the situation where both forces are prevalent but neither dominate. These waves are called Alfven-Inertial or Magnetic-Coriolis(MC) waves. We start with the Navier-Stokes equation which reduces to

$$\frac{\partial \mathbf{u}}{\partial t} + 2(\mathbf{\Omega}\times\mathbf{u}) = -\frac{1}{\rho_o}\nabla P + \frac{1}{\rho_o \mu}\left(\mathbf{B_o}\cdot\nabla\right)\mathbf{b}$$

Taking the curl gives

$$\frac{\partial(\nabla \times \mathbf{u})}{\partial t} + 2(\nabla \times (\boldsymbol{\Omega} \times \mathbf{u})) = -\frac{1}{\rho_o}(\nabla \times \nabla P) + \frac{1}{\rho_o \mu}\left(\mathbf{B_o} \cdot \nabla\right)(\nabla \times \mathbf{b})$$

$$\frac{\partial(\nabla \times \mathbf{u})}{\partial t} - 2(\boldsymbol{\Omega} \cdot \nabla)\mathbf{u} = \frac{1}{\rho_o \mu}\left(\mathbf{B_o} \cdot \nabla\right)(\nabla \times \mathbf{b})$$

Time derivative

$$\frac{\partial^2(\nabla \times \mathbf{u})}{\partial t^2} - 2(\boldsymbol{\Omega} \cdot \nabla)\frac{\partial \mathbf{u}}{\partial t} = \frac{1}{\rho_o \mu}\left(\mathbf{B_o} \cdot \nabla\right)\frac{\partial(\nabla \times \mathbf{b})}{\partial t}$$

$$\frac{\partial^2(\nabla \times \mathbf{u})}{\partial t^2} - 2(\boldsymbol{\Omega} \cdot \nabla)\frac{\partial \mathbf{u}}{\partial t} = \frac{1}{\rho_o \mu}\left(\mathbf{B_o} \cdot \nabla\right)^2(\nabla \times \mathbf{u})$$

Taking the curl one more time and simplifying

$$\frac{\partial^2(\nabla \times \nabla \times \mathbf{u})}{\partial t^2} + 2(\boldsymbol{\Omega} \cdot \nabla)\frac{\partial(\nabla \times \mathbf{u})}{\partial t} = \frac{1}{\rho_o \mu}\left(\mathbf{B_o} \cdot \nabla\right)^2(\nabla \times \nabla \times \mathbf{u})$$

$$\frac{\partial(\nabla \times \mathbf{u})}{\partial t} = -\frac{\nabla^2 \mathbf{u}}{2(\boldsymbol{\Omega} \cdot \nabla)}\left(\frac{\partial^2}{\partial t^2} - \frac{1}{\rho_o \mu}\left(\mathbf{B}_o \cdot \nabla\right)^2\right)$$

Taking a time derivative of ( ) and replacing the time derivative of the curl of the fluid parcel velocity into it

$$\frac{\partial^3(\nabla \times \mathbf{u})}{\partial t^3} - 2(\boldsymbol{\Omega} \cdot \nabla)\frac{\partial^2 \mathbf{u}}{\partial t^2} = \frac{1}{\rho_o \mu}\left(\mathbf{B_o} \cdot \nabla\right)^2 \frac{\partial(\nabla \times \mathbf{u})}{\partial t}$$

$$-\frac{1}{2(\boldsymbol{\Omega} \cdot \nabla)}\frac{\partial^2 \nabla^2 \mathbf{u}}{\partial t^2}\left(\frac{\partial^2}{\partial t^2} - \frac{1}{\rho_o \mu}\left(\mathbf{B}_o \cdot \nabla\right)^2\right) + \frac{1}{\rho_o \mu}\left(\mathbf{B}_o \cdot \nabla\right)^2 \frac{\nabla^2 \mathbf{u}}{2(\boldsymbol{\Omega} \cdot \nabla)}\left(\frac{\partial^2}{\partial t^2} - \frac{1}{\rho_o \mu}\left(\mathbf{B}_o \cdot \nabla\right)^2\right) = 2(\boldsymbol{\Omega} \cdot \nabla$$

$$-\frac{\nabla^2 \mathbf{u}}{2(\boldsymbol{\Omega} \cdot \nabla)}\left(\frac{\partial^2}{\partial t^2} - \frac{1}{\rho_o \mu}\left(\mathbf{B}_o \cdot \nabla\right)^2\right)\left(\frac{\partial^2}{\partial t^2} - \frac{1}{\rho_o \mu}\left(\mathbf{B}_o \cdot \nabla\right)^2\right) = 2(\boldsymbol{\Omega} \cdot \nabla)\frac{\partial^2 \mathbf{u}}{\partial t^2}$$

Simplifying yields a wave equation for an MC wave

$$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{\rho_o \mu}\left(\mathbf{B}_o \cdot \nabla\right)^2\right)^2 \nabla^2 \mathbf{u} = -4(\boldsymbol{\Omega} \cdot \nabla)^2 \frac{\partial^2 \mathbf{u}}{\partial t^2}$$

Finally I derive a wave equation for a Magnetic-Archimedes-Coriolis (MAC) wave. This wave equation takes into account the buoyant force arising from a temperature gradient. We will assume slow fluid motions so the time derivative of the velocity of the fluid parcel is ignored. The governing equations including the Navier-Stokes equation, magnetic induction equation and the heat equation simplify to

$$2(\mathbf{\Omega} \times \mathbf{u}) = -\frac{1}{\rho_o}\nabla P + \frac{1}{\mu\rho_o}\left(\mathbf{B}_o \cdot \nabla\right)\mathbf{b} + g\alpha\Theta\widehat{\mathbf{z}}$$
$$\frac{\partial \mathbf{b}}{\partial t} = \left(\mathbf{B_o} \cdot \nabla\right)\mathbf{u}$$
$$\frac{\partial \Theta}{\partial t} = \beta\mathbf{u}_z$$

After taking a time derivative and curl of the Navier Stokes equation and then inserting the time derivative of the the curl of the induction equation we operate again taking the curl and multiplying by $\left(B_0 \cdot \nabla\right)^2/\mu\rho_o$. Then we dot this equation with the z unit vector and replace the time derivative of the temperature perturbation by its respective value above. Rearranging we get a wave equation in the z-velocity of the fluid parcel.

$$\left(4(\mathbf{\Omega} \cdot \nabla)^2\frac{\partial^2}{\partial t^2} + \left[\frac{\left(\mathbf{B}_o \cdot \nabla\right)^2}{\mu\rho_o}\right]^2\nabla^2 - g\alpha\beta\frac{\left(\mathbf{B}_o \cdot \nabla\right)^2}{\mu\rho_o}\nabla_H^2\right)\mathbf{u}_z = 0$$

Results:

To better understand these wave equations I assume a plane wave solution of the form

$$\mathbf{u} = \mathrm{Re}\left\{\widehat{\mathbf{u}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right\}$$

After implementing this solution in each of these equations above, I solve for the angular frequency of the waves and receive a dispersion relation for the various waves. A dispersion relation relates a waves frequency with its wavelength or in my case the wave number which is $2\pi/$ wavelength (λ). Using the dispersion relation and making a few assumptions I can then solve for various wave properties such as period and wave velocity.

Inertial Wave:

$$\omega^2 = \frac{4(\mathbf{\Omega} \cdot \mathbf{k})^2}{k^2}$$

From this we see that inertial waves cannot exist perpendicular to the rotation axis in this plane geometry $(\mathbf{\Omega} \cdot \mathbf{k}) = 0$. We also see that the speed of propagation is highly influenced by the angular velocity of rotation which would be expected. Wave velocity $= \omega/\mathbf{k}$. Interestingly we see that the inertial wave does not have a dependence on the density of the fluid. In the special case where the wave propagates parallel to the rotation axis

$$\omega = 2\sqrt{\Omega} \quad T_{IW} = \frac{\pi}{\sqrt{\Omega}} \quad v_{p,IW} = \frac{2\sqrt{\Omega}}{k} = \frac{\lambda\sqrt{\Omega}}{\pi}$$

Alfven Wave:

$$\omega^2 = \frac{(\mathbf{B}_o \cdot \mathbf{k})^2}{\mu\rho_o}$$

Alfven waves in our geometry of an infinite plane cannot propagate perpendicular to the background magnetic field. Their angular frequency is proportional to their wavelength (Larger wavelength, larger angular frequency) and the strength of the background magnetic field. We also see they have dependence on the density of the fluid. With higher densities, the slower the angular frequency of the wave which matches my intuition. In the special case where the wave propagates parallel to the background magnetic field

$$\omega = \frac{B_o k}{\sqrt{\mu\rho_o}} \quad T_{AW} = \frac{2\pi\sqrt{\mu\rho_o}}{B_o k} \quad v_{p,AW} = \frac{B_o}{\sqrt{\mu\rho_o}}$$

MC Wave:

$$\omega \simeq \frac{k\,(\mathbf{B}_o \cdot \mathbf{k})^2}{2(\mathbf{\Omega} \cdot \mathbf{k})\mu\rho_o}$$

In the special case where the wave travels parallel to the rotation axis and the magnetic field is parallel to the rotation axis we find.

We see that shorter wavelengths increase the velocity of propagation. We also see that

$$\omega = \frac{k^2 B_o^2}{2\Omega\mu\rho_o} \quad T_{MC} = \frac{4\pi\Omega\mu\rho_o}{k^2 B_o^2} \quad v_{p,MC} = \frac{kB_o^2}{2\Omega\mu\rho_o} = \frac{\pi B_o^2}{\lambda\Omega\mu\rho_o}$$

the magnetic field strength plays a large role in the frequency, period, and wave velocity. Increased rotation rate/density will slow the MC wave.

MAC Wave:

$$\omega = \pm\frac{k\left(\mathbf{B}_o \cdot \mathbf{k}\right)^2}{2(\mathbf{\Omega}\cdot\mathbf{k})\mu\rho_o}\left(1 + \frac{g\alpha\beta\rho_o\mu\left(k_x^2 + k_y^2\right)}{k^2\left(\mathbf{B}_o \cdot \mathbf{k}\right)^2}\right)^{1/2} = \pm\frac{\omega_M^2}{\omega_C}\left(1 + \frac{\omega_A^2}{\omega_M^2}\right)^{1/2}$$

Just like our other waves we see that the MAC wave cannot propagate normal to the background magnetic field or normal to the rotation axis however they can propagate across field lines. We also see that if the buoyancy forces are neglected the dispersion relation reduces to that of an MC wave, which we should expect.

3 Discussion

The angular dispersion relations I calculated are an important stepping stone in realizing and conceptualizing what wave motions are possible within a electrically conducting fluid. These wave motions are most likely present within the outer core of the Earth however the dispersion relations derived above would not give an accurate description due to the simplified nature of the calculation. You can see from the results that the complexity of these wave motions ramps up quite quickly as more forces are incorporated in the force balance. My derivation is a relatively simple one, for example my calculations are diffusionless and do not take into account magnetic or viscous diffusivity. I also assume an infinite plane geometry which radically simplifies the problem compared to the actual spherical shell geometry of the outer core. In this geometry the coriolis effect varies with latitude and the beta plane approximation can be used to incorporate this effect (Gill, 1982; Pedlosky, 1987). Other geometries can be used in trying to elucidate wave motions from electrically conducting fluids in a rotating frame. A popular model is the quasi-geostrophic annulus model which is essentially a cylindrical shell with a sloped top(Busse (1976) and Soward (1979)).

This geometry unveils much more plausible wave motions that could possibly be present within the outer core.

4 Conclusion

While my calculations may not be directly applicable to the geometry of the outer core of the Earth, they do reveal important motions that are present in all dynamic electrically conducting fluids. Incorporating more terms to the force balance and including different geometries will allow calculations to become more representative to actual hydromagnetic waves within the outer core of the Earth. Further development of numerical models of these waves and comparison with geomagnetic observations data will provide greater evidence for their role in the secular variation of the geomagnetic field.

References

Alexandrescu, Mioara, et al. "Detection of geomagnetic jerks using wavelet analysis." Journal of Geophysical Research: Solid Earth 100.B7 (1995): 12557-12572.

Alexandrescu, Mioara, et al. "Worldwide wavelet analysis of geomagnetic jerks." Journal of Geophysical Research: Solid Earth 101.B10 (1996): 21975-21994.

Le Huy, Minh, et al. "On the characteristics of successive geomagnetic jerks." Earth, planets and space 50.9 (1998): 723-732.

Bloxham, Jeremy, Stephen Zatman, and Mathieu Dumberry. "The origin of geomagnetic jerks." Nature 420.6911 (2002): 65-68.

Chambodut, Aude, et al. "Wavelet frames: an alternative to spherical harmonic representation of potential fields." Geophysical Journal International 163.3 (2005): 875-899.

De Michelis, Paola, and Roberta Tozzi. "A local intermittency measure (LIM) approach to the detection of geomagnetic jerks." Earth and Planetary Science Letters 235.1 (2005): 261-272.

Malkin, Z. "Free core nutation and geomagnetic jerks." Journal of Geodynamics 72 (2013): 53-58.

Malkin, Zinovy. "Free core nutation: New large disturbance and connection evidence with geomagnetic jerks." arXiv preprint arXiv:1603.03176 (2016). 1961.

Chandrasekhar, S. Hydrodynamic and hydromagnetic stability. Clarendon Press,

Braginsky, S. I. Magnetohydrodynamics of the Earth's core. Geomagnetism and Aeronomy, 7, 698-712, 1964.

Braginsky, S. I. Magnetic waves in the Earth's core. Geomagnetism and Aeronomy, 7, 851-859, 1967.

Hide, R. Free hydromagnetic oscillations of the Earth's core and the theory of geomagnetic secular variation. Phil. Trans. R. Soc. Lond. A, 259, 615-647, 1966.

Hide, R. Motions of the Earth's core and mantle and variations in the main geomagnetic field. Science, 157, 55-56, 1967.

Malkus, W. V. R. Hydromagnetic planetary waves. J. Fluid Mech., 28(4), 793-802, 1967.

Melchoir, P. Physics of the Earth's core. Pergamon Press, 1986

Moffatt, H. K. Magnetic field generation in electrically conducting fluids. Cambridge Univeristy Press, 1978.

Finlay, Christopher Charles. Hydromagnetic waves in Earth's core and their influence on geomagnetic secular variation. Diss. University of Leeds, 2005.

Davidson, P.A. An introduction to magnetohydrodynamics. Cambridge Univeristy Press, 2001.

Gill, A. E. Atmosphere-Ocean dynamics. Cambridge Univeristy Press, 1982.

Pedlosky, J. Geophysical fluid dynamics. Cambridge Univeristy Press, 1987.

Busse, F. H. Asymptotic theory of convection in a rotating, cylindrical annulus. J. Fluid Mech., 173, 545-556, 1986.

Soward, A. M. Convection driven dynamos. Phys. Earth Planet. Int., 20, 134-151, 1979