FUNCTIONAL RENORMILIZATION GROUP

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1 Functional Calculus

Before we continue on to functional renormalization group methods we must first develop the formal mathematics and algebra of functionals.

1.1 What is a functional?

A ***function*** is a mathematical machine which relates elements between two sets. The function maps an element x or elements $\mathbf{x} \equiv \{x_1, x_2, \cdots, x_n\}$ from the first set to a single element in the second set.

$$f(\mathbf{x}) \mapsto \mathbb{R}, \mathbb{C}, \mathbb{Z}, \mathbb{R}^{m\times n}, \cdots$$

A ***functional*** is a mathematical machine which maps a *function* $f(\mathbf{x})$ or set of functions $\mathbf{f}(\mathbf{x}) \equiv \{f_1(\mathbf{x}), f_2(\mathbf{x}), \cdots, f_n(\mathbf{x})\}$ to a real or complex number.

$$F[\mathbf{f}(\mathbf{x})] \mapsto \mathbb{R}, \mathbb{C} \tag{1.1}$$

The functional $F[f(x)]$ is dependent on all the values of f at all values of x on the domain, an infinite number of independent variables!

A simple example

$$F[f(x)] = \int_0^1 f(x)^2 dx \tag{1.2}$$

$$F[x^2] = \frac{1}{5} \qquad F[\sin(x)] = \frac{1}{2} - \frac{\sin(2)}{4} \approx 0.27268 \tag{1.3}$$

Consider a function $F(\{x_n\})$ of a discrete set of N evenly spaced points on the interval $[a, b]$. In the limit where the spacing $\epsilon \to 0$ and $N \to \infty$, $\{x_n\} \to f(x)$ and $F(\{x_n\}) \to F[f(x)]$. Upon varying $\{x_n\}$ the function will change according to

$$dF(\{x_n\}) = \sum_{n=1}^{N} \left.\frac{\partial F}{\partial x_n}\right|_{x^0} dx_n \tag{1.4}$$

Noting the definition of an integral,

$$\int dx f(x) = \lim_{\epsilon\to 0} \sum_{n=1}^{N} \epsilon f(x_n) \tag{1.5}$$

We rewrite Eq.(1.4)

$$= \sum_{n=1}^{N} \epsilon \left(\frac{1}{\epsilon} \frac{\partial F}{\partial x_n}\bigg|_{x^0} \right) dx_n \tag{1.6}$$

In the limit that $\epsilon \to 0$ we introduce the notation $dx_n \to \delta f(x)$ to denote the infinitesimal variation of the function $f(x)$ and Eq.(1.6) becomes

$$dF[f(x)] = \int_a^b dx \frac{\delta F[f(x)]}{\delta f(x)}\bigg|_{f^0(x)} \delta f(x) \tag{1.7}$$

This tells us that the change in the functional is a sum of terms proportional to the infinitesimal change $\delta f(x)$ of the function and the constant of proportionality is the *functional derivative* $\delta F/\delta f(x)$.

Lets do a quick example illustrating how to calculate the functional derivative using the functional defined above, namely $F[f(x)] = \int_0^1 dx f(x)^2$. First we need to calculate the change in $F[f(x)]$ due to an infinitesimal change $\delta f(x)$

$$F[f(x) + \delta f(x)] = \int_0^1 \left[f(x^2) + 2f(x)\delta f(x) + \mathcal{O}(\delta f(x)^2) \right] dx \tag{1.8}$$

$$dF[f(x)] = F[f(x) + \delta f(x)] - F[f(x)] = \int_0^1 2f(x)\delta f(x) dx \tag{1.9}$$

Comparing with Eq.(1.7) we see that the functional derivative is just

$$\frac{\delta F[f(x)]}{\delta f(x)} = 2f(x) \tag{1.10}$$

Unfortunately the functionals we typically deal with aren't this simple but the calculation of the functional derivative is the same in every case.

2 Generating Functional $Z[J]$

2.1 Probability Theory

A useful analogy between the generating function of probability theory and the generating functional of quantum field theories warrants a brief appetizer before the main course.

We consider a continuous random variable ϕ whose outcomes are real numbers i.e. $\mathcal{S}_\phi = \{-\infty < \phi < \infty\}$. With this random variable we associate a normalized probability distribution $p(\phi)$

$$\text{prob}(\mathcal{S}) = \int_{-\infty}^{\infty} p(\phi) d\phi = 1 \tag{2.1}$$

such that the expectation value of any function $f(\phi)$ of the random variable ϕ is given by

$$\langle f(\phi) \rangle = \int_{-\infty}^{\infty} f(\phi) p(\phi) d\phi \tag{2.2}$$

The moments of the probability distribution are expectation value of powers of the random variable. The nth moment is

$$m_n \equiv \langle \phi^n \rangle = \int_{-\infty}^{\infty} \phi^n p(\phi) d\phi \tag{2.3}$$

The ***generating function*** or ***moment generating function*** of a distribution is given by the Fourier transform of the probability density function

$$z(j) = \langle e^{j\phi} \rangle = \int_{-\infty}^{\infty} e^{j\phi} p(\phi) d\phi \tag{2.4}$$

The moments of the distribution can then be obtained via derivatives of the generating function

$$m_n = \left(\frac{\partial}{\partial j} \right)^n z(j) \bigg|_{j=0} \tag{2.5}$$

or by expanding $z(j)$ in powers of j.

$$z(j) = \left\langle \sum_{n=0}^{\infty} \frac{j^n}{n!} \phi^n \right\rangle = \sum_{n=0}^{\infty} \frac{j^n}{n!} \langle \phi^n \rangle \tag{2.6}$$

We can also define the ***cumulant generating function*** whose expansion generates the cumulants of the distribution defined as

$$w(j) = \ln z(j) = \sum_{n=1}^{\infty} \frac{j^n}{n!} \langle \phi^n \rangle_c \tag{2.7}$$

Where $\langle \phi^n \rangle_c \equiv c_n$ are the cumulants of the distribution and can also be obtained via derivatives of the cumulant generating function

$$c_n = \left(\frac{\partial}{\partial j} \right)^n w(j) \bigg|_{j=0} \tag{2.8}$$

The relationship between moments and cumulants can be found by equating the two expressions

$$z(j) = e^{w(j)} \tag{2.9}$$

$$\sum_{n=0}^{\infty} \frac{j^n}{n!}\langle\phi^n\rangle = \exp\left[\sum_{m=1}^{\infty} \frac{j^m}{m!}\langle\phi^m\rangle_c\right] \tag{2.10}$$

$$\sum_{n=0}^{\infty} \frac{j^n}{n!}\langle\phi^n\rangle = \prod_{m=1}^{\infty} \sum_{k_m=0}^{\infty} \left[\frac{1}{k_m!}\left(\frac{j^m\langle\phi^m\rangle_c}{m!}\right)^{k_m}\right] \tag{2.11}$$

Matching powers of j on both sides leads to the relation

$$\langle\phi^n\rangle = \sum_{\{k_m\}}{}' n! \prod_m \frac{1}{k_m!}\left(\frac{\langle\phi^m\rangle_c}{m!}\right)^{k_m} \tag{2.12}$$

Where the sum is restricted so that $\sum_m mk_m = n$. For example, if we are interested writing the third moment $\langle\phi^3\rangle$ in terms of cumulants the sum is constrained to values of k_m and m such that

$$\sum_m mk_m = 3 \tag{2.13}$$

$$k_1 + 2k_2 + 3k_3 + \cdots + mk_m = 3 \tag{2.14}$$

There are only three possible ways to satisfy this constraint.

$$\{\mathbf{k}\} = \begin{cases} k_1 = 3, k_2 = k_3 = \cdots = k_m = 0 \\ k_1 = k_2 = 1, k_3 = \cdots = k_m = 0 \\ k_3 = 1, k_1 = k_2 = \cdots = k_m = 0 \end{cases} \tag{2.15}$$

Eq.(2.12) becomes

$$\langle\phi^3\rangle = 3!\left[\frac{1}{3!}\frac{\langle\phi\rangle_c^3}{(1!)^3} + \frac{1}{1!}\frac{\langle\phi\rangle_c}{1!}\frac{1}{1!}\frac{\langle\phi^2\rangle_c}{2!} + \frac{1}{1!}\frac{\langle\phi^3\rangle_c}{3!}\right] \tag{2.16}$$

$$\langle\phi^3\rangle = \langle\phi\rangle_c^3 + 3\langle\phi\rangle_c\langle\phi^2\rangle_c + \langle\phi^3\rangle_c \tag{2.17}$$

By the same method, we also have

$$\begin{aligned} \langle\phi\rangle &= \langle\phi\rangle_c \\ \langle\phi^2\rangle &= \langle\phi^2\rangle_c + \langle\phi\rangle_c^2 \\ \langle\phi^4\rangle &= \langle\phi^4\rangle_c + 4\langle\phi^3\rangle_c\langle\phi\rangle_c + 6\langle\phi^2\rangle_c\langle\phi\rangle_c^2 + \langle\phi\rangle_c^4 \end{aligned} \tag{2.18}$$

Eq.(2.12) also has a graphical interpretation. Represent the mth cumulant as a *connected cluster* of m points. The nth moment is then obtained by summing all possible subdivisions of n points into groupings of smaller (connected or disconnected) clusters. The contribution of each subdivision to the sum is the product of the connected cumulants that it represents.

3 Time Evolution as a Path Integral

In quantum mechanics and quantum field theory, the time evolution of a quantum system is described by the unitary time-evolution operator.

$$\mathcal{U}(t) = e^{-i\mathcal{H}(t-t_0)} \tag{3.1}$$

Where t_0 is some reference time. In many cases, especially in quantum field theory we would like to calculate the probability amplitude of a particular process to occur in some time interval Δt, such as transitions between energy eigenstates or an interaction between two particles. In the context of field theories we are interested in the probability amplitude that our field beginning in some configuration ϕ_i at some time t_i ends up in some configuration ϕ_f at some later time t_f. Setting our reference time $t_0 = 0$ the amplitude can be expressed as

$$\langle \phi_f(\mathbf{x}, t_f)|\phi_i(\mathbf{x}, t_i)\rangle = \langle \phi_f(\mathbf{x})|e^{-i\mathcal{H}\Delta t}|\phi_i(\mathbf{x})\rangle \tag{3.2}$$

In the case of scalar fields and adopting the path integral formalism we can also express this amplitude as a weighted sum over all possible field configurations. Symbolically we write

$$\langle \phi_f(\mathbf{x})|e^{-i\mathcal{H}\Delta t}|\phi_i(\mathbf{x})\rangle = \int \mathcal{D}[\phi] \exp\left[i\int_0^{\Delta t} d^4x\mathcal{L}\right] \tag{3.3}$$

in which the functions $\phi(x)$ which we integrate over are constrained to the particular configuration ϕ_i and ϕ_f at times $t = t_i$ and $t = t_f$ respectively.

We will be interested in Lagrangian systems which include “source” terms of the form

$$\int d^4xF(\phi)J(x) \tag{3.4}$$

Where $F(\phi)$ represents some arbitrary function of the fields and J is the “source” field. In the context of quantum field theories we can view this time-dependent source as a means of spontaneous particle creation. In practice this source term will prove as a useful construct when we go to write a generating functional of time-ordered $n-$point correlation functions in the context of the path integral formulation.

We assume that the source $J(x)$ will be non-zero only in a finite interval $t \in [T_1, T_2]$ and take $t_2 > T_2$, $t_1 < T_1$. We then compute the vacuum to vacuum amplitude. That is, given that our field configuration minimizes the total energy functional at time $t_1 \to -\infty$ what is the amplitude that the field will remain in this configuration at time $t_2 \to \infty$? We can compute this quantity by considering general initial $\phi_a(x)$ and final $\phi_b(x)$ field configurations.

$$\langle \phi_b(\mathbf{x}, t_2) | \phi_a(\mathbf{x}, t_1) \rangle_J = \int \mathcal{D}[\phi] \exp\left[i \int_{t_1}^{t_2} d^4x \mathcal{L}(\phi, \partial_\mu \phi, J) \right] \tag{3.5}$$

Where the J reminds us that we are dealing with a transition in the presence of a source J. The integration over field configurations can be factored into a product of transition amplitudes corresponding to before, during, and after the source is turned on

$$\int \mathcal{D}[\phi] = \langle \phi_b | e^{i\mathcal{H}(t_2 - t_1)} | \phi_a \rangle \tag{3.6}$$

Using the completeness relation $\mathbf{1} = \int \mathcal{D}[\phi] \, |\phi\rangle \, \langle \phi|$

$$= \int \mathcal{D}[\phi_c]\mathcal{D}[\phi_d] \, \langle \phi_b(\mathbf{x}) | e^{-i\mathcal{H}(t_2 - T_2)} | \phi_d(\mathbf{x}) \rangle \, \langle \phi_d(\mathbf{x}, T_2) | \phi_c(\mathbf{x}, T_1) \rangle_J \, \langle \phi_c(\mathbf{x}) | e^{-i\mathcal{H}(T_1 - t_1)} | \phi_a(\mathbf{x}) \rangle \tag{3.7}$$

Inserting a complete set of energy eigenstates

$$\begin{aligned} = \sum_{m,n} \int \mathcal{D}[\phi_c]\mathcal{D}[\phi_d] \, \langle \phi_b(\mathbf{x}) | e^{-i\mathcal{H}t_2} | m \rangle \, \langle m | e^{i\mathcal{H}T_2} | \phi_d(\mathbf{x}) \rangle \, \langle \phi_d(\mathbf{x}, T_2) | \phi_c(\mathbf{x}, T_1) \rangle_J \\ \times \langle \phi_c(\mathbf{x}) | e^{-i\mathcal{H}T_1} | n \rangle \, \langle n | e^{i\mathcal{H}t_1} | \phi_a(\mathbf{x}) \rangle \end{aligned} \tag{3.8}$$

$$\begin{aligned} = \sum_{m,n} e^{-i(E_m t_2 - E_n t_1)} \, \langle \phi_b(\mathbf{x}) | m \rangle \, \langle n | \phi_a(\mathbf{x}) \rangle \int \mathcal{D}[\phi_c]\mathcal{D}[\phi_d] \, \langle m | e^{i\mathcal{H}T_2} | \phi_d(\mathbf{x}) \rangle \\ \times \langle \phi_d(\mathbf{x}, T_2) | \phi_c(\mathbf{x}, T_1) \rangle_J \, \langle \phi_c(\mathbf{x}) | e^{-i\mathcal{H}T_1} | n \rangle \end{aligned} \tag{3.9}$$

$$\begin{aligned} = \sum_{m,n} e^{-i(E_m t_2 - E_n t_1)} \phi_{b,m}(\mathbf{x}) \phi^*_{a,n}(\mathbf{x}) \\ \times \int \mathcal{D}[\phi_c]\mathcal{D}[\phi_d] \phi^*_{d,m}(\mathbf{x}, T_2) \, \langle \phi_d(\mathbf{x}, T_2) | \phi_d(\mathbf{x}, T_1) \rangle_J \, \phi_{c,n}(\mathbf{x}, T_1) \end{aligned} \tag{3.10}$$

The above expression gives us a clear interpretation of the amplitude. The integral in the last line can be thought of as a field $\phi_{c,n}(\mathbf{x}, T_1)$, that is propagated through time when the source is acting via $\langle \phi_d(\mathbf{x}, T_2)|\phi_c(\mathbf{x}, T_1)\rangle_J$ and then dotted with the field $\phi^*_{d,m}(\mathbf{x}, T_2)$. The fields $\phi_{c,n}(\mathbf{x}, T_1)$ and $\phi^*_{d,m}(\mathbf{x}, T_2)$ are only energy eigenstates for times before and after the source, respectively. Thus, the integral is the probability amplitude that an energy eigenstate $|n\rangle$ will become an energy eigenstate $|m\rangle$ through the action of the source J. Now we would like to project out the ground state of our theory. To do this we can perform a rotation of the time-axis into the imaginary plane by some small angle $-\delta$ ($\delta > 0$). Under such a transformation

$$t_1 \to t_1 + i|t_1|\delta \tag{3.11}$$

$$t_2 \to t_2 - i|t_2|\delta \tag{3.12}$$

in which we have chosen our axis of rotation to lie between t_1 and t_2. We see that the exponential term

$$e^{-i(E_m t_2 - E_n t_1)} \to e^{-i(E_m(t_2 - i|t_2|) - E_n(t_1 + i|t_1|))} = e^{-\delta(E_m|t_2| + E_n|t_1|)} e^{-i(E_m t_2 - E_n t_1)} \tag{3.13}$$

acquires a damping that goes as $e^{-\delta(E_m|t_2|+E_n|t_1|)}$. In the limit which $t_1 \to -\infty$ and $t_2 \to \infty$ the exponential damping dies the slowest for $n = 0$ and we are left with

$$\begin{aligned}\lim_{t\to\infty e^{-i\delta}} \langle \phi_b(\mathbf{x}, t_2)|\phi_a(\mathbf{x}, t_1)\rangle_J =& e^{-iE_0(t_2-t_1)}\phi_{b,0}(\mathbf{x})\phi^*_{a,0}(\mathbf{x}) \\ &\times \int \mathcal{D}[\phi_c]\mathcal{D}[\phi_d]\phi^*_{d,0}(\mathbf{x}, T_2)\, \langle \phi_d(\mathbf{x}, T_2)|\phi_d(\mathbf{x}, T_1)\rangle_J\, \phi_{c,0}(\mathbf{x}, T_1)\end{aligned} \tag{3.14}$$

Where $t \equiv t_2 = -t_1$. In the limit where T_2 and $T_1 \to \pm\infty$ respectively, the integral reduces to the amplitude that a field configuration of the form $\phi_0(\mathbf{x})$ in the distant past will remain in the form $\phi_0(\mathbf{X})$ in the distant future. Put another way, it is the vacuum-to-vacuum transition amplitude in the presence of the source J denoted as

$$\langle \Omega|\Omega\rangle_J \propto \lim_{t\to\infty e^{-i\delta}} \langle \phi_b(\mathbf{x}, t)|\phi_a(\mathbf{x}, -t)\rangle_J \tag{3.15}$$

Instead of rotating the contour of the time integration we could have also added a small perturbation $-i\epsilon\phi^2(x)/2$ to our Hamiltonian. First order perturbation theory tells us that this will shift the energy levels by and amount $-i\epsilon \langle n|\phi^2|n\rangle /2$. Assuming that the expectation value of ϕ^2 increases with energy we see that this has the same effect as rotating the time axis in projecting out the vacuum state! Subtracting $i\epsilon\phi^2/2$ from the Hamiltonian is the same as adding $i\epsilon\phi^2/2$ to the Lagrangian. Thus,

$$\langle \Omega|\Omega\rangle_J \propto \int \mathcal{D}[\phi] \exp\left[i \int_{-\infty}^{\infty} dt \left\{ \mathcal{L}(\phi, \partial_\mu\phi) + F(\phi(x))J(x) + \frac{1}{2}i\epsilon\phi^2(x) \right\}\right] \tag{3.16}$$

We also want to normalize our result such that when $J \to 0$ the amplitude $\langle \Omega | \Omega \rangle = 1$. We thus define the following functional

$$Z[J] = \frac{\int \mathcal{D}[\phi] \exp\left[i \int_{-\infty}^{\infty} dt \left\{ \mathcal{L}(\phi, \partial_\mu \phi) + F(\phi(x))J(x) + \frac{1}{2} i\epsilon\phi^2(x) \right\} \right]}{\int \mathcal{D}[\phi] \exp\left[i \int_{-\infty}^{\infty} dt \left\{ \mathcal{L}(\phi, \partial_\mu \phi) + \frac{1}{2} i\epsilon\phi^2(x) \right\} \right]} \tag{3.17}$$

4 Generating Functional: Scalar Field Theory

As a nice example, consider a one-component real scalar field theory in $d-$dimensions. Now, rather than a single random variable ϕ, we deal with an *infinite dimensional* set of real random variables corresponding to the field amplitudes of the scalar field throughout $d-$dimensional spacetime $\mathcal{S}_{\phi(\mathbf{X})} = \{-\infty < \phi(\mathbf{X}_1), \phi(\mathbf{X}_2), \ldots, \phi(\mathbf{X}_{N\to\infty}) < \infty\}$ where each $\mathbf{X}$ corresponds to a $d-$dimensional vector corresponding to a point in spacetime $\mathbf{X} \equiv (x^1, x^2, x^3, \ldots, x^{d-1}, t)$. To identify and enumerate field configurations in practice you can imagine discretizing spacetime into N^d points separated by some uniform distance ϵ (or equivalently transforming spacetime into hypercubes of volume ϵ^d) and each labeled by their respective coordinate $(x_i^1, x_j^2, \ldots, x_k^{d-1}, t_l)$ for $i, j, k, \ldots, l = 1, 2, \ldots, N$. For calculational convenience we can then concatenate these indicies in a super-index n running from $n = 1, \ldots, N^d$. In the limiting process where $\epsilon \to 0$ and $N \to \infty$ we obtain our continuous quantum field $\phi(\mathbf{X})$. You can imagine on the discretized lattice each field configuration could then be identified by summing over all possible values of the field at each of the N^d lattice points iteratively. The limiting factor of this type of computation comes down to available memory and computing power. But, for a sufficiently dense lattice, these types of computations can be done rather efficiently and can produce useful physical results.

Given that the field amplitudes are continuous random variables, we can associate a *joint* probability distribution representing the probability density of a field configuration in a volume element $d^N\phi(\mathbf{X}) = \prod_{i=1}^N d\phi(X_i)$ around the point $\phi(\mathbf{X}) = \{\phi(\mathbf{X}_1), \phi(\mathbf{X}_2), \ldots, \phi(\mathbf{X}_{N\to\infty})\}$. In the context of quantum mechanics, we associate a distribution of *probability amplitudes* $P(\phi(\mathbf{X}))$ in field configuration space. Given some operator $\mathcal{O}(\phi(\mathbf{X}))$ that is a function of the field (i.e. field configurations) we can define its operator average or expectation value as

$$\langle \mathcal{O} \rangle = \mathcal{N} \int \mathcal{D}[\phi] P(\phi) \mathcal{O}(\phi) \tag{4.1}$$

The main goal of any quantum field theory is to produce $n-$point correlation functions.

The analogue to generating functional of disconnected correlators is defined as

$$Z[J] \equiv \int \mathcal{D}[\Phi] \exp\left[i \int d^4x \left[\mathcal{L}(\Phi, \partial_\mu \Phi) + J(x)\Phi(x)\right]\right] \tag{4.2}$$

The n-point correlation function of the theory can obtained via n functional derivatives of $Z[\Phi]$.

4.1 Free Scalar Field

Take, for example, the free Klein-Gordon theory in which the generating functional is given by

$$Z[J] = \int \mathcal{D}[\phi] \exp\left[i \int d^4x \left[\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + J\phi\right]\right] \tag{4.3}$$

The two-point Green's function is then given by

$$\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle \equiv \langle \phi\phi \rangle = \frac{1}{Z[0]}\left[-i\frac{\delta}{\delta J(x_1)}\right]\left[-i\frac{\delta}{\delta J(x_2)}\right] Z[J]\Bigg|_{J=0} \tag{4.4}$$

In the free theory the above expression can be calculated explicitly, we integrate our Lagrangian density by parts[1] to expose the Klein-Gordon operator

$$Z[J] = \int \mathcal{D}[\phi] \exp\left[i \int d^4x \left[-\frac{1}{2}\phi(\partial_\mu^2 + m^2)\phi + J\phi\right]\right] \tag{4.5}$$

And then perform a redefinition of the field $\phi \to \phi' + (\partial^2 + m^2)^{-1}J$. Plugging this into our generating functional

$$Z[J] = \mathcal{J} \int \mathcal{D}[\phi'] \exp\left[i \int d^4x \left[-\frac{1}{2}\phi'(\partial^2 + m^2)\phi' + \frac{1}{2}J(\partial^2 + m^2)^{-1}J\right]\right] \tag{4.6}$$

Where $\mathcal{J}$ is the Jacobian of the transformation. The Jacobian is trivial to compute as long as one remembers that

$$\int \mathcal{D}\phi \approx \lim_{i\to\infty} \prod_i \int d\phi(x_i) \to \begin{vmatrix} \frac{\partial\phi(x_1)}{\partial\phi'(x_1)} & \frac{\partial\phi(x_1)}{\partial\phi'(x_2)} & \cdots & \frac{\partial\phi(x_1)}{\partial\phi'(x_n)} \\ \frac{\partial\phi(x_2)}{\partial\phi'(x_1)} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{\partial\phi(x_n)}{\partial\phi'(x_1)} & \frac{\partial\phi(x_n)}{\partial\phi'(x_2)} & \cdots & \frac{\partial\phi(x_n)}{\partial\phi'(x_n)} \end{vmatrix} \prod_i \int d\phi'(x_i) \tag{4.7}$$

[1] And assume our real scalar field vanishes at infinity

For our transformation, or in general for translations of the fields, the Jacobian matrix is equivalent to the n-dimensional identity matrix

$$\frac{\partial\phi(x_i)}{\partial\phi'(x_j)} = \frac{\partial}{\partial\phi'(x_j)}\left[\phi'(x_i) + (\partial^2+m^2)^{-1}J\right] = \frac{\partial\phi'(x_i)}{\partial\phi'(x_j)} = \delta_{ij} \tag{4.8}$$

And thus, the Jacobian for the transformation is 1 due to the fact that

$$\det[\mathbb{1}_{n\times n}] = 1 \tag{4.9}$$

Thus, we see that our generating functional can be rewritten as

$$Z[J] = \int \mathcal{D}[\phi'] \exp\left[-i\int d^4x \frac{1}{2}\phi'(\partial^2+m^2)\phi'\right] \exp\left[i\int d^4x\frac{1}{2}J(\partial^2+m^2)^{-1}J\right] \tag{4.10}$$

$$= Z[0]\exp\left[\frac{i}{2}\int d^4xJ(\partial^2+m^2)^{-1}J\right] \tag{4.11}$$

Using the fact that $(\partial^2+m^2)^{-1}$ is just the Green's function of the Klein-Gordon operator D_F

$$= Z[0]\exp\left[-\frac{1}{2}\int d^4xd^4yJ(x)D_F(x-y)J(y)\right] \tag{4.12}$$

Plugging into our expression for the two-point function

$$\langle\phi\phi\rangle = -\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\exp\left[-\frac{1}{2}\int d^4xd^4yJ(x)D_F(x-y)J(y)\right]\Bigg|_{J=0} \tag{4.13}$$

Explicitly we have

$$\begin{aligned} = -\frac{\delta}{\delta J(x_1)}&\left[-\frac{1}{2}\int d^4xd^4y\delta^{(4)}(x-x_2)D_F(x-y)J(y) - \int d^4xd^4y\delta^{(4)}(y-x_2)J(x)D_F(x-y)\right] \\ &\times\exp\left[-\frac{1}{2}\int d^4xd^4yJ(x)D_F(x-y)J(y)\right]\Bigg|_{J=0} \end{aligned} \tag{4.14}$$

$$\begin{aligned} = \frac{\delta}{\delta J(x_1)}&\left[\frac{1}{2}\int d^4yD_F(x_2-y)J(y) + \frac{1}{2}\int d^4xJ(x)D_F(x-x_2)\right] \\ &\times\exp\left[-\frac{1}{2}\int d^4xd^4yJ(x)D_F(x-y)J(y)\right]\Bigg|_{J=0} \end{aligned} \tag{4.15}$$

We only need to find terms which do not contain J as we are to set these terms equal to zero anyway. The only terms which survive are the terms in brackets from the first line of Eq.(4.15) leaving

$$\langle\phi\phi\rangle = \frac{1}{2}D_F(x_2-x_1) + \frac{1}{2}D_F(x_2-x_1) = D_F(x_2-x_1) \tag{4.16}$$

Likewise we can also compute the four-point function, but first we will concatenate our notation for calculational brevity. Let the arguments be denoted as subscripts, $\phi_1 \equiv \phi(x_1)$, $D_{xy} \equiv D_F(x-y)$, $J_x \equiv J(x)$, etc. And let any repeated indices be integrated over, the four-point function is given by

$$\begin{aligned}
\langle T\{\phi_1\phi_2\phi_3\phi_4\}\rangle &= (-i)^4 \frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\frac{\delta}{\delta J_4} \exp\left[-\frac{1}{2}J_x D_{xy} J_y\right]\Bigg|_{J=0} \\
&= \frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\left[-\frac{1}{2}D_{4y}J_y - \frac{1}{2}J_x D_{x4}\right]\exp\left[-\frac{1}{2}J_x D_{xy} J_y\right]\Bigg|_{J=0} \\
&= \frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\left[-J_x D_{x4}\right]\exp\left[-\frac{1}{2}J_x D_{xy} J_y\right]\Bigg|_{J=0} \\
&= \frac{\delta}{\delta J_1}\left[D_{34}J_x D_{x2} + D_{24}J_y D_{y3} + J_x D_{x4} D_{23}\right]\exp\left[-\frac{1}{2}J_x D_{xy} J_y\right]\Bigg|_{J=0} \\
&= D_{34}D_{12} + D_{24}D_{13} + D_{14}D_{23}
\end{aligned} \tag{4.17}$$

Where in the first term of the third line we rename $y \to x$ and use the fact that $D_{4x} = D_{x4}$.

In the free theory, the functional derivatives act on the generating functional and produce all combinations of connections between pairs of $n-$points to form the $n-$point correlation function. While the above was a convenient demonstration of how to explicitly calculate the $n-$point correlation function, it is only useful for the free-theory in which we can express the generating functional in the form of Eq.(4.13). We could have also calculated the form of the two point function directly from Eq.(4.4)

$$\langle\phi\phi\rangle = \frac{1}{Z[0]}\left[-i\frac{\delta}{\delta J(x_1)}\right]\left[-i\frac{\delta}{\delta J(x_2)}\right]\int \mathcal{D}[\phi]\exp\left[i\int d^4x\left[-\frac{1}{2}\phi(\partial_\mu^2 + m^2)\phi + J\phi\right]\right]\Bigg|_{J=0} \tag{4.18}$$

$$\langle\phi\phi\rangle = \frac{\int \mathcal{D}[\phi]\phi(x_2)\phi(x_1)\exp\left[i\int d^4x\left[-\frac{1}{2}\phi(\partial_\mu^2 + m^2)\phi\right]\right]}{\int \mathcal{D}[\phi]\exp\left[i\int d^4x\left[-\frac{1}{2}\phi(\partial_\mu^2 + m^2)\phi\right]\right]} \tag{4.19}$$

5 Functional Methods in QFT

5.1 Dyson-Schwinger Equations

In quantum field theories all physical content is stored in correlation functions.

$$\langle\phi_i(x_1)\phi_j(x_2)\cdots\phi_k(x_n)\rangle = \int \mathcal{D}\phi_i(x_1)\phi_j(x_2)\cdots\phi_k(x_n)e^{-S[\phi_i]} \tag{5.1}$$

The 'usual' calculation of n-point correlation functions involves a perturbative expansion powers of an infinitesimal parameter. Of course this procedure fails when we have to deal with couplings which are no longer $\ll 1$. In these cases we need to turn to non-perturbative methods to calculate correlation functions. One such method resulting in a non-linear first-order functional differential equation was developed by Dyson and Schwinger. The core entity in these calculations is the effective action $\Gamma[\Phi_i]$. The effective action is defined as

$$\Gamma[\Phi] \equiv \sup_J \left(\int [-W[J] + \Phi_i J_i] \right) \tag{5.2}$$

Where the J_i's denote sources for the fields Φ_i and $W[J]$ is the generating functional of connected correlators. The generating functional is related to the bare action $S[\phi]$ via the path integral

$$Z[J] \equiv e^{W[J]} = \int \mathcal{D}[\phi] e^{-S[\phi] + \phi_j J_j} \tag{5.3}$$

All n-point correlation functions can be generated by taking functional derivatives of the generating functional with respect to the sources.

$$\langle \phi_i(x_1)\phi_j(x_2)\cdots\phi_k(x_n)\rangle = \frac{1}{Z[0]} \left[\frac{\delta^n Z[J]}{\delta J_i(x_1)\delta J_j(x_2)\cdots\delta J_k(x_n)} \right]_{J=0} \tag{5.4}$$

Where ϕ_i denotes the quantum fields. At J_{sup} the variation of the effective action must be zero and thus,

$$\frac{\delta \Gamma_{\text{sup}}}{\delta J_j(x)} = 0 = \frac{\delta}{\delta J_j(x)} \left(\int -W[J] + \Phi_i J_i \right) \tag{5.5}$$

$$\frac{\delta W[J]}{\delta J_j(x)} = \Phi_i \delta_{ij} \tag{5.6}$$

But,

$$\frac{\delta W[J]}{\delta J_j(x)} = \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J_j(x)} = \langle \phi(x) \rangle_J \tag{5.7}$$

Thus the average field Φ is given by

$$\Phi_i \equiv \langle \phi_i \rangle_J = \frac{\delta W}{\delta J_i} = Z[J]^{-1} \int \mathcal{D}[\phi] \phi_i e^{-S[\phi] + \phi_i J_i} \tag{5.8}$$

Noting that

$$\frac{\delta \Gamma_{\text{sup}}[\Phi]}{\delta \Phi_i(x)} = -\int \frac{\delta W}{\delta J_j(y)} \frac{\delta J_j(y)}{\delta \Phi_i(x)} dy + \int \frac{\delta \Phi_k(y)}{\delta \Phi_i(x)} J_k dy + \int \frac{\delta J_l(y)}{\delta \Phi_i(x)} \Phi_l dy \tag{5.9}$$

$$= -\Phi_j\delta_{ji} + J_k\delta_{ki} + \Phi_l\delta_{li} = J_i(x) \tag{5.10}$$

We can now write down a functional differential equation for the effective action considering Eq.(5.3)

$$e^{-\Gamma[\Phi]} = \int \mathcal{D}[\phi] \exp\left[-S[\phi+\Phi] + \frac{\delta\Gamma[\Phi]}{\delta\Phi_i}\phi_i\right] \tag{5.11}$$

An exact solution for $\Gamma[\Phi]$ is difficult to obtain but we can perform a vertex expansion of $\Gamma[\Phi]$

$$\Gamma[\Phi] = \sum_{n=0}^{\infty} \frac{1}{\mathcal{N}^{i_1\cdots i_n}} \sum_{i_1\cdots i_n} \int d^Dx_1 \cdots d^Dx_n \Gamma^{i_1\cdots i_n} \Phi_{i_1}(x_1)\cdots\Phi_{i_n}(x_n) \tag{5.12}$$

Where $\mathcal{N}$ is the corresponding symmetry factor and $\Gamma^{i_1\cdots i_n}$ correspond to the one-particle irreducible proper vertices. Inserting this back into Eq.(5.11) and comparing the coefficients of the field monomials results in an infinite tower of coupled integro-differential equations for $\Gamma^{i_n\cdots i_n}$, these are the Dyson-Schwinger equations.

5.2 Functional Renormalization Group Flows

The core entity in functional renormalization group is the ***scale dependent*** effective average action $\Gamma_k[\Phi_i]$ where k is the scaling parameter. The effective action interpolates between a microscopic UV description for the bare action at some scale $k = \Lambda$ and a macroscopic description at low energies described by the full quantum action, with k=0. The scale parameter k acts as an infrared regulator suppressing any quantum fluctuations with momentum less than k. This allows us to study how the parameters of our theory 'flow' from high energies to low energies and vice versa. Below our goal is to construct an equation to describe the flow of the effective average action for **bosonic** degrees of freedom. Generalizations to fermionic and gauge degrees of freedom will be discussed in later sections.

We now define an IR regulated generating functional

$$e^{W_k[J]} \equiv Z_k[J] = e^{-\Delta S_k\left[\frac{\delta}{\delta J}\right]} Z[J] = \int \mathcal{D}[\phi] e^{-S[\phi]-\Delta S_k[\phi]+\int J_i\phi_i} \tag{5.13}$$

Where

$$\Delta S_k[\phi] = \frac{1}{2}\phi_i R_k^{ij}\phi_j = \frac{1}{2}\int \frac{d^Dq}{(2\pi)^D}\frac{d^Dq'}{(2\pi)^D} R_k^{ab}(q,q')\phi_a(q)\phi_b(q') \tag{5.14}$$

With $R(q,q') \equiv R(q)\delta^D(q-q)$

$$= \frac{1}{2}\int \frac{d^Dq}{(2\pi)^D} R_k^{ab}(q)\phi_a(q)\phi_b(-q) \tag{5.15}$$

Where the regulator function R_k must satisfy the following properties

1. $\lim_{q^2/k^2 \to 0} R_k > 0$ (implements IR regularization for the path integral)
2. $\lim_{k^2/q^2 \to 0} R_k = 0$ (regulator must vanish for $k \to 0$)
3. $\lim_{k \to \Lambda \to \infty} R_k = \infty$ (functional integral is then dominated by the stationary point of the action)

There are many regulator functions which satisfy these requirements. A detailed discussion is coming...

The effective average action is then defined as

$$\Gamma_k[\Phi] = -W_k[J] + \int_{x/p}\left[J_i\Phi_i - \frac{1}{2}\Phi_i R_k^{ij}\Phi_j\right] \tag{5.16}$$

The quantum equations of motion can also be obtained by taking functional derivatives with respect to the fields

$$\begin{aligned}\frac{\delta\Gamma_k[\Phi_i]}{\delta\Phi_i(x)} = &-\int \frac{\delta W_k}{\delta J_j(y)}\frac{\delta J_j(y)}{\delta\Phi_i(x)}dy + \int\frac{\delta\Phi_k(y)}{\delta\Phi_i(x)}J_k dy + \int\frac{\delta J_l(y)}{\delta\Phi_i(x)}\Phi_l dy \\ &-\frac{1}{2}\int\frac{\delta(\Phi_m(y)\Phi_m(y))}{\delta\Phi_i(x)}R_k^{mm}(y) - \frac{1}{2}\int\frac{\delta(\Phi_p(y)\Phi_q(y))}{\delta\Phi_i(x)}R_k^{pq}(y)\end{aligned} \tag{5.17}$$

Where $p \neq q$

$$= \delta(x-y)\left[-\Phi_j\delta_{ji} + J_k\delta_{ki} + \Phi_l\delta_{li} - \delta_{im}\Phi_{im}R - \frac{1}{2}\left(\delta_{ip}\Phi_q + \delta_{iq}\Phi_p\right)R\right] \tag{5.18}$$

For a single scalar field this reduces to

$$\frac{\delta\Gamma_k[\Phi]}{\delta\Phi(x)} = J(x) - \Phi(x)R_k(x) \tag{5.19}$$

or

$$J(x) = \frac{\delta\Gamma_k[\Phi]}{\delta\Phi(x)} + \Phi(x)R_k(x) \tag{5.20}$$

Thus,

$$\frac{\delta J(x)}{\delta\Phi(y)} = \frac{\delta^2\Gamma_k[\Phi]}{\delta\Phi(x)\delta\Phi(y)} + \frac{\delta\Phi(x)}{\delta\Phi(y)}R_k(x) \tag{5.21}$$

$$= \frac{\delta^2\Gamma_k[\Phi]}{\delta\Phi(x)\delta\Phi(y)} + \delta(x-y)R_k(x) \tag{5.22}$$

But, from Eq.(5.8)

$$\frac{\delta\Phi(x)}{\delta J(z)} = \frac{\delta^2 W_k[J]}{\delta J(x)\delta J(z)} \equiv G_k(x-z) \tag{5.23}$$

Where $G_k(x-z)$ is the *connected* correlator

$$G_k(p) = \frac{\delta^2 W_k}{\delta J\delta J} = \langle\phi(-p)\phi(p)\rangle - \langle\phi(-p)\rangle\langle\phi(p)\rangle \tag{5.24}$$

Using these relations we have the following identity

$$\delta(x-y) = \frac{\delta J(x)}{\delta J(y)} = \int d^d y \frac{\delta J(x)}{\delta\Phi(z)}\frac{\delta\Phi(z)}{\delta J(y)} = \int d^d y\left[\frac{\delta^2\Gamma_k[\Phi]}{\delta\Phi(x)\delta\Phi(z)} + R_k(z)\right]G_k(z-y) \tag{5.25}$$

Introducing the notation

$$\Gamma_k^{(n)}[\phi] = \frac{\delta^n\Gamma_k[\Phi]}{\delta\Phi(x_1)\delta\Phi(x_2)\cdots\delta\Phi(x_n)} \tag{5.26}$$

As a matrix equation we have

$$\mathbb{1} = [\Gamma_k^{(2)} + R_k]G_k \tag{5.27}$$

Or

$$[\Gamma_k^{(2)} + R_k]^{-1} = G_k \tag{5.28}$$

Taking the derivative of $W_k[J_i]$ with respect to t at J_{sup} for a fixed source (k-independent)

$$\partial_t W_k = k\frac{d}{dk}\ln Z_k = k\frac{1}{Z_k}\frac{d}{dk}Z_k = -\partial_t\int\mathcal{D}[\phi]\Delta S_k[\phi] = -\partial_t\langle\Delta S_k[\phi]\rangle \tag{5.29}$$

$$= -\frac{1}{2}\int\mathcal{D}[\phi]\int\frac{d^d q}{(2\pi)^d}\partial_t R_k^{ab}\phi_a(-q)\phi_b(q) \tag{5.30}$$

$$= -\frac{1}{2}\int\frac{d^d q}{(2\pi)^d}\partial_t R_k^{ab}\int\mathcal{D}[\phi]\phi_a(-q)\phi_b(q) \tag{5.31}$$

$$= -\frac{1}{2}\int \frac{d^d q}{(2\pi)^d}\partial_t R_k^{ab}\langle\phi_a(-q)\phi_b(q)\rangle \tag{5.32}$$

From Eq.(5.24)

$$= -\frac{1}{2}\int \frac{d^d q}{(2\pi)^d}\partial_t R_k\left[\frac{\delta^2 W_k[J]}{\delta J_a(-q)\delta J_b(q)} + \langle\phi_a(-q)\rangle\langle\phi_b(q)\rangle\right] \tag{5.33}$$

$$= -\frac{1}{2}\int \frac{d^d q}{(2\pi)^d}\left[\partial_t R_k G_k + \Phi_a(-q)\partial_t R_k^{ab}\Phi_b(q)\right] \tag{5.34}$$

To put everything together we take the scale-derivative of Eq.(5.16)

$$\partial_t\Gamma_k = -\partial_t W_k + \int_p\left[\partial_t(J_i\Phi_i) - \frac{1}{2}\Phi_a(\partial_t R_k^{ab})\Phi_b\right] \tag{5.35}$$

$$\partial_t\Gamma_k = -\partial_t W_k - \int_p \frac{1}{2}\Phi_a(\partial_t R_k^{ab})\Phi_b \tag{5.36}$$

$$\partial_t\Gamma_k = \int \frac{d^d q}{(2\pi)^d}\left[\frac{1}{2}\partial_t R_k G_k + \frac{1}{2}\Phi_a\partial_t(R_k^{ab})\Phi_b - \frac{1}{2}\Phi_a(\partial_t R_k^{ab})\Phi_b\right] \tag{5.37}$$

$$\partial_t\Gamma = \frac{1}{2}\int \frac{d^d q}{(2\pi)^d}\partial_t R_k G_k \tag{5.38}$$

Using Eq.(5.28) we can rewrite G_k

$$\partial_t\Gamma_k = \frac{1}{2}\int_q \partial_t R_k\left[\Gamma_k^{(2)} + R_k\right]^{-1} \tag{5.39}$$

This functional differential equation can be written independent of a basis as

$$\partial_k\Gamma_k[\Phi] = \frac{1}{2}\left[(\Gamma_k[\Phi] + R_k^{-1}\right]^{ji}\partial_k R_k^{ij} \tag{5.40}$$

$$\partial_k\Gamma_k[\Phi] = \frac{1}{2}\mathrm{Tr}\left[(\Gamma_k[\Phi] + R_k)^{-1}\partial_k R_k\right] \tag{5.41}$$

The flow equations for mass terms, coupling constants, wave-function renormalization, etc. are extracted from the flow equations for the n-point correlation functions. To obtain them we just take functional derivatives with respect to the fields. In doing so we end up with another infinite tower of coupled functional differential equations.

To express in a particular representation we insert the complete set of states, for example in position space the trace is written as

$$\mathrm{Tr}[\hat{O}] = \sum_i \langle \Psi_i | \hat{O} | \Psi_i \rangle = \sum_i \int d^d x d^d y \, \langle \Psi_i | x \rangle \langle x | \hat{O} | y \rangle \langle y | \Psi_i \rangle \tag{5.42}$$

$$= \int d^d x d^d y \, \langle x | \hat{O} | y \rangle \sum_i \Psi_i^*(x) \Psi_i(y) = \int d^d x d^d y \, \langle x | \hat{O} | y \rangle \, \delta^d(x-y) = \int d^d x \, \langle x | \hat{O} | x \rangle \tag{5.43}$$

$$\equiv \int d^d x \hat{O}(x) \tag{5.44}$$

Sometimes it is convenient to work in a momentum basis, performing a Fourier transform from the position-basis

$$\int d^d x \, \langle x | \hat{O} | x \rangle = \int d^d x d^d p d^d p' \, \langle x | p \rangle \langle p | \hat{O} | p' \rangle \langle p' | x \rangle \tag{5.45}$$

$$= \frac{1}{(2\pi)^d} \int d^d x d^d p d^d p' e^{i(p-p')\cdot x} \langle p | \hat{O} | p' \rangle \tag{5.46}$$

$$= \frac{1}{(2\pi)^d} \int d^d p d^d p' (2\pi)^d \delta^d(p-p') \langle p | \hat{O} | p' \rangle = \int d^d p \, \langle p | \hat{O} | p \rangle \tag{5.47}$$

$$\equiv \int d^d p \hat{O}(p) \tag{5.48}$$

6 O(1) Scalar field theory

6.1 ϕ^4 theory β-functions in d-dimensions

The effective average action of an $O(1)$ scalar field theory with potential $V(\Phi^2)$ is given by

$$\Gamma_k[\Phi] = \int d^d x \left[\frac{1}{2} \partial^\mu \Phi(x) \partial_\mu \Phi(x) + V_k(\Phi^2) \right] \tag{6.1}$$

Introducing the notation $V_k'(\Phi^2) \equiv \delta V_k(\Phi^2)/\delta \Phi^2$ we have

$$\Gamma_k^{(2)}[\Phi] = -\partial^2 + 2V_k'(\Phi^2) + 4\Phi^2(x) V''(\Phi^2) \tag{6.2}$$

Plugging our result into the Wetterich equation we have

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \mathrm{Tr} \left[\frac{\partial_t R_k}{[-\partial^2 + 2V_k' + 4\Phi^2(x) V'' + R_k]} \right] \tag{6.3}$$

The trace involves integration of position and momentum space

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \int d^d x \int \frac{d^d p}{(2\pi)^4} \left[\frac{\partial_t R_k}{[p^2 + 2V_k' + 4\Phi^2(x) V'' + R_k]} \right] \tag{6.4}$$

$$= \frac{1}{(4\pi)^{d/2}\Gamma(d/2)} \int d^d x \int_0^\infty d\tilde{p} \tilde{p}^{d/2-1} \left[\frac{\partial_t R_k}{[\tilde{p}^2 + 2V_k' + 4\Phi^2(x) V'' + R_k]} \right] \tag{6.5}$$

Where $\Gamma(d/2)$ refers to Euler's gamma function and $\tilde{p} \equiv |p|$. Plugging in Litim's regulator noting that $\partial_t R_k^L(p) = 2k^2 \Theta(k^2 - p^2)$

$$\partial_t \Gamma_k[\Phi] = \frac{2k^2}{(4\pi)^{d/2}\Gamma(d/2)} \int d^d x \int_0^{k^2} d\tilde{p} \tilde{p}^{d/2-1} \left[\frac{1}{k^2 + 2V_k' + 4\Phi^2(x) V''} \right] \tag{6.6}$$

$$\partial_t \Gamma_k[\Phi] = \frac{2}{d} \frac{1}{(4\pi)^{d/2}\Gamma(d/2)} \left[\frac{1}{[k^2 + 2V_k' + 4\Phi^2(x) V'']} \right] \tag{6.7}$$

If we consider stationary field configurations the kinetic term drops out and we are left with

$$\partial_t V_k(\Phi^2) = \frac{2}{d} \frac{1}{(4\pi)^{d/2}\Gamma(d/2)} \left[\frac{1}{[k^2 + 2V_k' + 4\Phi^2(x) V'']} \right] \tag{6.8}$$

To analyze the problem further we need to make an explicit choice for the potential. One choice is to take a Taylor expansion in powers of $\rho \equiv \Phi^2$

$$V(\rho) = \sum_{n=1}^{N} \lambda_{2n} \rho^n \tag{6.9}$$

The couplings can by extracted from the exact renormalization group flow equation via

$$\lambda_{2n} = \frac{1}{n!} \frac{\partial^n V}{\partial \rho^n} \bigg|_{\rho=0} \tag{6.10}$$

Likewise, the β-functions can then be obtained via

$$\beta_{2n} = \partial_t \lambda_{2n} = \frac{1}{n!} \frac{\partial^n}{\partial \rho^n} \partial_t V \bigg|_{\rho=0} \tag{6.11}$$

Now we redefine the couplings and fields to obtain the dimensionless β-functions

$$\bar{\Phi} = k^{\frac{2-d}{2}} \Phi \tag{6.12}$$

$$\bar{\lambda} = k^{(d-2)n-d} \lambda_{2n} \tag{6.13}$$

$$\partial_t \bar{\lambda_{2n}} = ((d-2)n - d)\bar{\lambda_{2n}} + k^{(d-2)n-d}\beta_{2n} \tag{6.14}$$

$$\bar{V}_k(\bar{\rho}) = k^{-d} V_k(\rho) \tag{6.15}$$

Eq.(6.8) becomes

$$\partial_t \bar{V}_k = -d\bar{V} + (d-2)\rho\bar{V}' + \frac{2}{d}\frac{1}{(4\pi)^{d/2}\Gamma(d/2)}\left[\frac{1}{1+2\bar{V}_k'+4\rho\bar{V}''}\right] \tag{6.16}$$

The equation above is the basis for what is done in `scalarFRG.py`. Obtaining the β-functions for the dimensionless couplings is now a matter of applying Eq.(6.11).

6.2 Fixed points

Ultimately the β-functions allow a way to non-pertubatively analyze the UV behavior of our theory. Of particular interest are are stationary or fixed points i.e. positions in theory space in which all β-functions are zero. The trivial solution corresponding to $\lambda_2 = \lambda_4 = \cdots = \lambda_{2n} = 0$ is called a Gaussian fixed point. These fixed points correspond to a free theory with no interactions. Non-trivial solutions, corresponding to renormalizable interacting theories, are called Wilson-Fisher fixed points and are the main motivation in the computing the $\beta-$functions.

7 Flow Plots from `scalarFRG.py`

With `scalarFRG` I am able to compute the β-functions up to an arbitrary number of terms in the expansion of the potential. For the flow plots I've just expanded up to two terms, if more terms are required it would be necessary to scan the parameters space of all included couplings in order to obtain a complete picture of the flows.

7.1 $d = 1, n = 2$

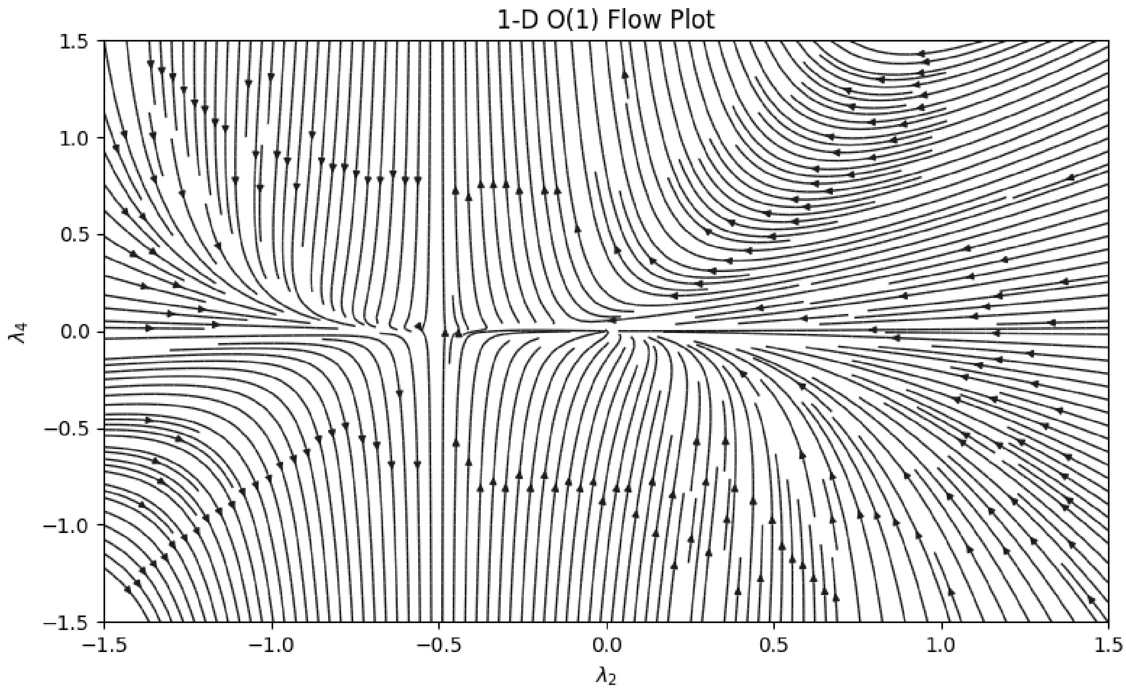

Figure 1: Flow for $d = 1, n = 2$

7.2 $d = 2, n = 2$

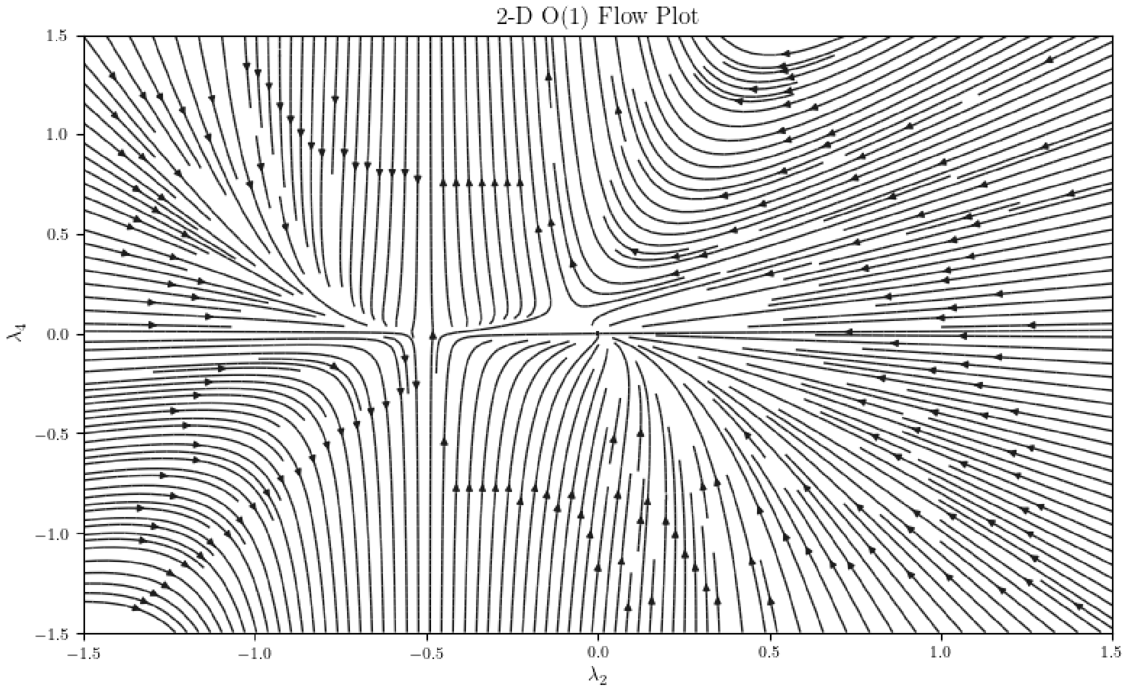

Figure 2: Flow for $d = 1, n = 2$

7.3 $d = 3, n = 2$

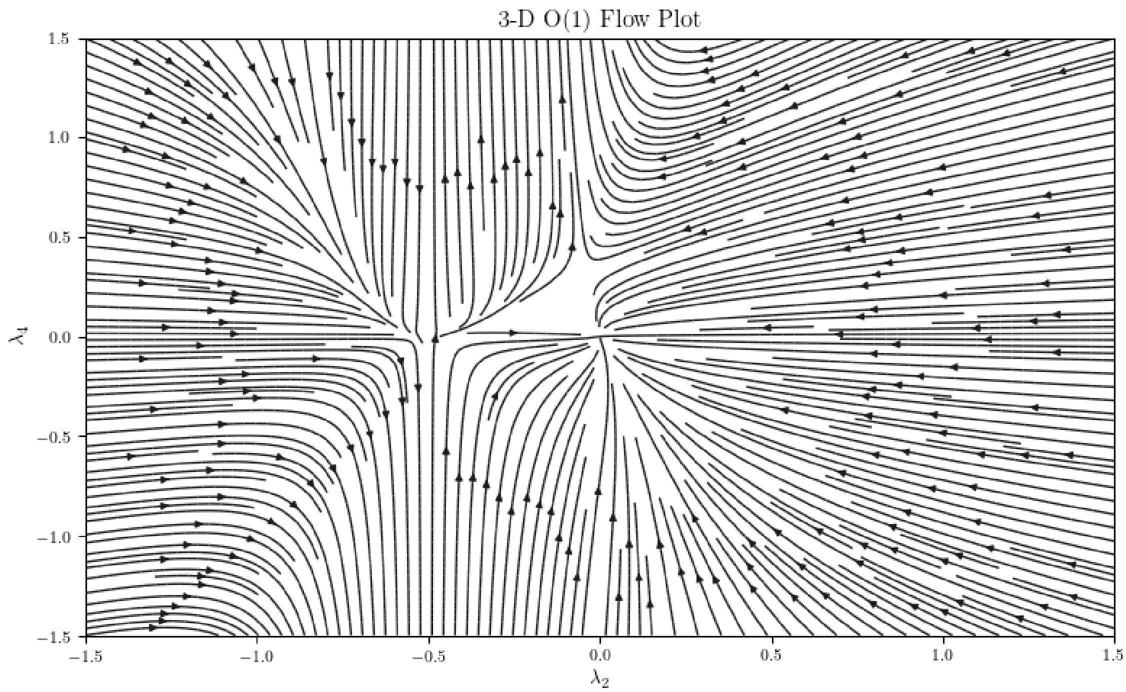

Figure 3: Flow for $d = 1, n = 2$

7.4 $d = 4, n = 2$

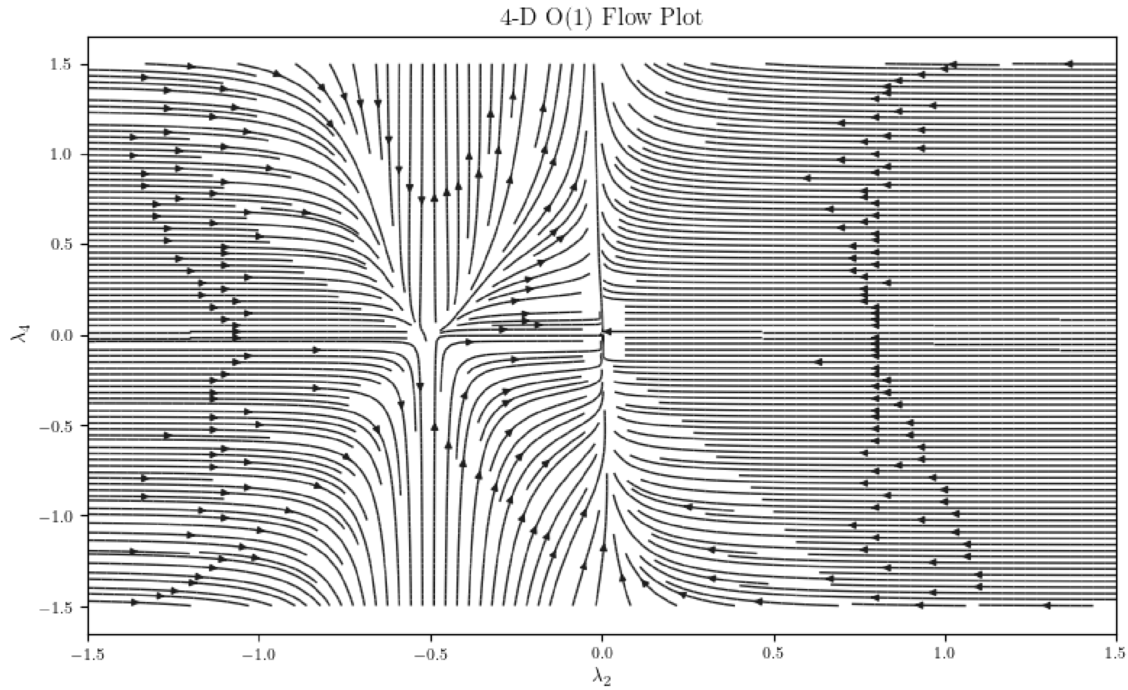

Figure 4: Flow for $d = 1, n = 2$

8 Detailed calculation

Truncating the potential at $n = 2$ we have the following effective average action

$$\Gamma_k[\Phi] = \int d^d x \left[\frac{1}{2} z_k g_{\mu\nu} \partial^\mu \Phi(x) \partial^\nu \Phi(x) + \frac{1}{2} m_k^2 \Phi^2(x) + \frac{\lambda_k}{4!} \Phi^4(x) \right] \tag{8.1}$$

Taking a scale derivative of the average effective action yields

$$\partial_t \Gamma_k[\phi] = \int d^d x \left[\frac{1}{2} (\partial_t z_k) \partial^\mu \Phi(x) \partial_\mu \Phi(x) + \frac{1}{2} (\partial_t m_k^2) \Phi^2(x) + \frac{(\partial_t \lambda_k)}{4!} \Phi^4(x) \right] \tag{8.2}$$

We can project out the coupling β-functions via the projecting operator

$$\Pi_{(l,m)} \partial_t \Gamma_k[\Phi] = \frac{1}{l!} \frac{1}{m!} \partial_{\Phi_c}^l \partial_q^m \left(\partial_t \Gamma_k[\Phi_c e^{iq \cdot x}] \right) \Big|_{\Phi_c = 0, q = 0} \tag{8.3}$$

For example, to project out $\partial_t z_k$ we will need $\Pi_{(2,2)}$ going term by term for explicitness

$$-\int d^dx \frac{1}{2!}\frac{1}{2!}\frac{1}{2}\partial_t z_k \partial^2_{\Phi_c}\partial^2_q \left[\partial_\mu(\Phi_c e^{iq\cdot x})\partial^\mu(\Phi_c e^{iq\cdot x})\right]\Big|_{\Phi_c=0,q=0} \tag{8.4}$$

$$= \frac{-(i)^2}{8}\int d^dx \partial_t z_k \partial^2_{\Phi_c}\partial^2_q \left[q^2\Phi_c^2 e^{2iq\cdot x}\right]\Big|_{\Phi_c=0,q=0} \tag{8.5}$$

$$= \frac{1}{8}\int d^dx \partial_t z_k \partial^2_{\Phi_c}\left[2\Phi_c^2 e^{2iq\cdot x} + 8ixq\Phi_c^2 e^{2iq\cdot x} + (2ix)^2 q^2 \Phi_c^2 e^{2iq\cdot x}\right]\Big|_{\Phi_c=0,q=0} \tag{8.6}$$

$$= \frac{1}{8}\int d^dx \partial_t z_k \partial^2_{\Phi_c}\left[2\Phi_c^2\right]\Big|_{\Phi_c=0} \tag{8.7}$$

$$= \frac{1}{2}\int d^dx \partial_t z_k \tag{8.8}$$

For the next term we have

$$\int d^dx \frac{1}{2!}\frac{1}{2!}\frac{1}{2}\partial_t m_k^2 \partial^2_{\Phi_c}\partial^2_q \left[(\Phi_c^2 e^{2iq\cdot x})\right]\Big|_{\Phi_c=0,q=0} = \frac{1}{8}\int d^dx \partial_t m_k^2 \partial^2_{\Phi_c}[\Phi_c^2(2ix)^2]\Big|_{\Phi_c=0} \tag{8.9}$$

$$= -\int d^dx x^2 \partial_t m_k^2 \tag{8.10}$$

And finally for the last term

$$\int d^dx \frac{1}{2!}\frac{1}{2!}\frac{1}{4!}\partial_t \lambda_k \partial^2_q \partial^2_{\Phi_c}\left[\Phi_c^4 e^{4iq\cdot x}\right]\Big|_{\Phi_c=0,q=0} \tag{8.11}$$

$$= \frac{1}{96}\int d^dx \partial_t \lambda_k \partial^2_q \left[(4\cdot 3)\Phi_c^2 e^{4iq\cdot x}\right]\Big|_{\Phi_c=0,q=0} = 0 \tag{8.12}$$

Putting it all together we find

$$\Pi_{(2,2)}\partial_t \Gamma_k[\Phi] = \int d^dx \left[\frac{1}{2}\partial_t z_k - x^2 \partial_t m_k^2\right] \tag{8.13}$$

The other parameters can be found easily by acting with $\Pi_{(2,0)}$ and $\Pi_{(4,0)}$, we are left with the following three results

$$\Pi_{(2,2)}\partial_t \Gamma_k[\Phi] = \int d^dx \left[\frac{1}{2}\partial_t z_k - x^2 \partial_t m_k^2\right] \tag{8.14}$$

$$\Pi_{(2,0)}\partial_t\Gamma_k[\Phi] = \frac{1}{2}\int d^dx\partial_t m_k^2 \tag{8.15}$$

$$\Pi_{(4,0)}\partial_t\Gamma_k[\Phi] = \frac{1}{4!}\int d^dx\partial_t\lambda_k \tag{8.16}$$

Now we look to the RHS of the Wetterich equation

$$\frac{1}{2}\mathrm{Tr}\left[(\Gamma_k^{(2)}[\Phi] + R_k)^{-1}\partial_t R_k\right] = \tag{8.17}$$

First we need to compute $\Gamma_k^{(2)}[\Phi]$

$$\begin{aligned}\frac{\delta^2\Gamma_k}{\delta\Phi(y)\delta\Phi(z)} = \int d^dx\Bigg\{&\frac{\delta^2}{\delta\Phi(y)\delta\Phi(z)}\left[\frac{1}{2}z_k\partial^\mu\Phi(x)\partial_\mu\Phi(x)\right] + \frac{\delta^2}{\delta\Phi(y)\delta\Phi(z)}\left[\frac{1}{2}m_k^2\Phi^2(x)\right]\\ &+ \frac{\delta^2}{\delta\Phi(y)\delta\Phi(z)}\left[\frac{\lambda_k}{4!}\Phi^4(x)\right]\Bigg\}\end{aligned} \tag{8.18}$$

$$= \int d^dx\left\{\frac{1}{2}\frac{\delta^2}{\delta\Phi(y)\delta\Phi(z)}\left[\Phi(x)\left(-z_k\partial^2 + m_k^2\right)\Phi(x)\right] + \frac{1}{4!}\lambda_k\frac{\delta^2}{\delta\Phi(y)\delta\Phi(z)}\Phi^4(x)\right\} \tag{8.19}$$

For the first term I have integrated by parts (with the assumption that our field $\Phi(x)$ vanishes at the boundaries),

$$\int d^dx[\partial^\mu\Phi\partial_\mu\Phi] = -\int d^dx\Phi\partial^2\Phi \tag{8.20}$$

Defining $\Delta \equiv -z_k\partial^2 + m_k^2$ the first term becomes

$$\frac{1}{2}\frac{\delta}{\delta\Phi(y)}\left[\frac{\delta}{\delta\Phi(z)}[\Phi(x)]\Delta\Phi(x) + \Phi(x)\Delta\frac{\delta}{\delta\Phi(z)}[\Phi(x)]\right] \tag{8.21}$$

$$= \frac{1}{2}\left[\Delta\frac{\delta}{\delta\Phi(y)}\Phi(z) + \frac{\delta}{\delta\Phi(y)}\Phi(z)\Delta\right] \tag{8.22}$$

$$= \Delta \tag{8.23}$$

The second term gives

$$\frac{1}{4!}\lambda_k\frac{\delta^2}{\delta\Phi(y)\delta\Phi(z)}\Phi^4(x) = \frac{4\cdot 3}{4!}\lambda_k\Phi^2(x) = \frac{1}{2}\lambda_k\Phi^2(x) \tag{8.24}$$

Putting this all together we are left with

$$\Gamma_k^{(2)}[\Phi] = -z_k\partial^2 + m_k^2 + \frac{1}{2}\lambda_k\Phi^2(x) \tag{8.25}$$

Now we can plug this into the scale evolution Eq.(**??**)

$$\partial_t\Gamma_k[\Phi] = \frac{1}{2}\mathrm{Tr}\left[\frac{1}{\left[-z_k\partial^2 + m_k^2 + \frac{\lambda_k}{2}\Phi^2(x) + R_k\right]}\partial_t R_k\right] \tag{8.26}$$

Now we define $\tilde{\Delta} \equiv -z_k\partial^2 + m_k^2 + R_k$

$$= \frac{1}{2}\mathrm{Tr}\left[\frac{1}{\tilde{\Delta}\left[\mathbb{1} + \frac{1}{2}\tilde{\Delta}^{-1}\lambda_k\Phi^2(x)\right]}\partial_t R_k\right] \tag{8.27}$$

Now I can make use of the following operator expansion[2]

$$\left[\mathbb{1} + \tilde{\Delta}^{-1}\lambda_k\Phi^2(x)\right]^{-1} = \sum_{k=0}^{\infty}\frac{(-1)^k}{2^k}\left[\tilde{\Delta}^{-1}\lambda_k\Phi^2(x)\right]^k \tag{8.28}$$

Truncating all operators above Φ^4 we are left with

$$= \frac{1}{2}\mathrm{Tr}\left[\tilde{\Delta}^{-1}\left\{\mathbb{1} - \frac{1}{2}\tilde{\Delta}^{-1}\lambda_k\Phi^2(x) + \frac{1}{4}\left(\tilde{\Delta}^{-1}\lambda_k\Phi^2(x)\right)^2 + \mathcal{O}(\Phi^6)\right\}\partial_t R_k\right] \tag{8.29}$$

$$= \frac{1}{2}\mathrm{Tr}\left[\left\{\tilde{\Delta}^{-1} - \frac{1}{2}(\tilde{\Delta}^{-1})^2\lambda_k\Phi^2(x) + \frac{1}{4}(\tilde{\Delta}^{-1})^3\lambda_k^2\Phi^4(x)\right\}\partial_t R_k\right] \tag{8.30}$$

Expanding the trace in momentum space ($\partial^2 \rightsquigarrow i^2p^2 = -p^2$)

$$\begin{aligned}\partial_t\Gamma_k[\Phi] = \frac{1}{2}\frac{1}{(2\pi)^d}\int d^dx\int d^dp\Bigg[\Bigg\{ &\left(z_kp^2 + m_k^2 + R_k(p)\right)^{-1} \\ &- \frac{\lambda_k}{2}\left(z_kp^2 + m_k^2 + R_k(p)\right)^{-2}\langle p|\Phi\Phi|p\rangle \\ &+ \frac{\lambda_k^2}{4}\left(z_kp^2 + m_k^2 + R_k(p)\right)^{-3}\langle p|\Phi\Phi\Phi\Phi|p\rangle\Bigg\}\partial_t R_k\Bigg]\end{aligned} \tag{8.31}$$

[2] For this expansion to be valid the eigenvalues λ_i of the operator $\tilde{\Delta}^{-1}\lambda_k\Phi^2$ must satisfy the condition $|\lambda_i| < 1$.

$$= \frac{1}{2}\frac{1}{(2\pi)^d}\int d^dx \int d^dp \Bigg[\Bigg\{ \left(z_k p^2 + m_k^2 + R_k(p)\right)^{-1} - \frac{\lambda_k}{2}\left(z_k p^2 + m_k^2 + R_k(p)\right)^{-2}\Phi^2(p) + \frac{\lambda_k^2}{4}\left(z_k p^2 + m_k^2 + R_k(p)\right)^{-3}\Phi^4(p)\Bigg\}\partial_t R_k\Bigg] \tag{8.32}$$

Applying the same projection operators (Eq.(8.3)) to this expression we are left with

$$\Pi_{(2,2)}\partial_t\Gamma_k[\Phi] = \frac{1}{(2\pi)^d}\int d^dx \int d^dp \frac{\lambda_k}{2}\left[\left(z_k p^2 + m_k^2 + R_k\right)^{-2} x^2 \partial_t R_k\right] \tag{8.33}$$

$$\Pi_{(2,0)}\partial_t\Gamma_k[\Phi] = -\frac{1}{(2\pi)^d}\int d^dx \int d^dp \frac{\lambda_k}{4}\left[(z_k p^2 + m_k^2 + R_k)^{-2}\partial_t R_k\right] \tag{8.34}$$

$$\Pi_{(4,0)}\partial_t\Gamma_k[\Phi] = \frac{1}{(2\pi)^d}\int d^dx \int d^dp \frac{\lambda_k^2}{8}\left[(z_k p^2 + m_k^2 + R_k)^{-3}\partial_t R_k\right] \tag{8.35}$$

Equating with Eqs.(8.14)(8.15)(8.16) we find the following three relations

$$\frac{1}{2}\partial_t z_k - x^2\partial_t m_k^2 = \frac{1}{(2\pi)^d}\int d^dp \frac{\lambda_k}{2}\left[\left(z_k p^2 + m_k^2 + R_k\right)^{-2} x^2 \partial_t R_k\right] \tag{8.36}$$

$$\frac{1}{2}\partial_t m_k^2 = -\frac{1}{(2\pi)^d}\int d^dp \frac{\lambda_k}{4}\left[(z_k p^2 + m_k^2 + R_k)^{-2}\partial_t R_k\right] \tag{8.37}$$

$$\frac{1}{4!}\partial_t \lambda_k = \frac{1}{(2\pi)^d}\int d^dp \frac{\lambda_k^2}{8}\left[(z_k p^2 + m_k^2 + R_k)^{-3}\partial_t R_k\right] \tag{8.38}$$

Plugging the result for $\partial_t m_k$ in Eq.(8.34) into Eq.(8.33) we see that

$$\partial_t z_k = 0 \tag{8.39}$$

To make further progress we need to choose a regulator function R_k. To start we will look at the optimized Litim regulator [2]

$$R_k^L(p) = z_k\left(k^2 - p^2\right)\Theta\left(k^2 - p^2\right) \tag{8.40}$$

Where $\Theta(x)$ is the Heaviside step function. Thus,

$$\partial_t R_k^L = 2z_k k^2\Theta(k^2 - p^2) \tag{8.41}$$

Plugging this regulator back in to Eq.(8.37)

$$\partial_t m_k^2 = -\frac{\lambda_k}{2}\frac{1}{(2\pi)^d}\int d^d p \left[\left(z_k p^2 + m_k^2 + z_k(k^2-p^2)\Theta(k^2-p^2)\right)^{-2} 2z_k k^2\Theta(k^2-p^2)\right] \tag{8.42}$$

$$= -\frac{\lambda_k}{(2\pi)^d}\int_{p^2<k^2} d^d p \left[z_k p^2 + m_k^2 + z_k(k^2-p^2)\right]^{-2} z_k k^2 \tag{8.43}$$

$$= -\frac{\lambda_k}{(2\pi)^d}\int_{p^2<k^2} d^d p \left[\frac{z_k k^2}{(m_k^2 + z_k k^2)^2}\right] \tag{8.44}$$

Now the problem has been boiled down to computing the volume of d-dimensional sphere of radius k

$$\Omega_d(k) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} k^d \tag{8.45}$$

Where $\Gamma(x) = (x-1)!$ is the Euler gamma-function. Note

$$\Gamma(n+\frac{1}{2}) = \frac{(2n)!}{4^n n!}\sqrt{\pi} \qquad \text{for } n \in \mathbb{Z}^* \tag{8.46}$$

Thus we find

$$= -\frac{z_k\lambda_k}{(2\pi)^d}\left[\frac{k^2}{(m_k^2 + z_k k^2)^2}\right]\frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} k^d \tag{8.47}$$

Our second flow equation is then given by

$$\partial_t m_k^2 = \frac{-\lambda_k z_k}{(2\sqrt{\pi})^d \Gamma\left(\frac{d}{2}+1\right)}\frac{k^{2+d}}{(m_k^2 + z_k k^2)^2} \tag{8.48}$$

Plugging the regulator into Eq.(8.38) gives

$$\partial_t \lambda_k = \frac{3\lambda_k^2}{(2\pi)^d}\int d^d p \left[\left(z_k p^2 + m_k^2 + z_k(k^2-p^2)\Theta(k^2-p^2)\right)^{-3} 2z_k k^2\Theta(k^2-p^2)\right] \tag{8.49}$$

$$= \frac{6\lambda_k^2}{(2\pi)^d}\int_{p^2<k^2} d^d p \left[\left(z_k p^2 + m_k^2 + z_k(k^2-p^2)\right)^{-3} z_k k^2\right] \tag{8.50}$$

$$= \frac{6\lambda_k^2}{(2\pi)^d}\int_{p^2<k^2} d^d p \left[\frac{z_k k^2}{(m_k^2 + z_k k^2)^3}\right] \tag{8.51}$$

We are left with our final flow equation

$$\partial_t \lambda_k = \frac{6\lambda_k^2 z_k}{(2\sqrt{\pi})^d \Gamma\left(\frac{d}{2}+1\right)}\frac{k^{2+d}}{(m_k^2 + z_k k^2)^3} \tag{8.52}$$

We can perform a redefinition of the couplings to obtain dimensionless β-functions

$$\bar{\lambda}_k = k^{d-4}\lambda_k \longrightarrow \lambda_k = \bar{\lambda}_k k^{4-d} \tag{8.53}$$

$$\bar{m}_k^2 = k^{-2}m_k^2 \longrightarrow m_k^2 = \bar{m}_k^2 k^2 \tag{8.54}$$

We have

$$\partial_t \lambda_k = k\frac{d}{dk}(\bar{\lambda}_k k^{4-d}) = k\left((4-d)k^{3-d}\bar{\lambda}_k + k^{4-d}\frac{d}{dk}\bar{\lambda}_k\right) = k^4 k^{-d}\left((4-d)\bar{\lambda}_k + k\frac{d}{dk}\bar{\lambda}_k\right) \tag{8.55}$$

$$\frac{6\lambda_k^2 z_k}{(2\sqrt{\pi})^d\Gamma\left(\frac{d}{2}+1\right)}\frac{k^{2+d}}{(m_k^2+z_k k^2)^3} = \frac{6\bar{\lambda}_k^2 k^{8-2d}k^{2+d}z_k}{(2\sqrt{\pi})^d\Gamma\left(\frac{d}{2}+1\right)(\bar{m}_k^2 k^2 + z_k k^2)^3} \tag{8.56}$$

$$= \frac{6\bar{\lambda}_k^2 k^4 k^{-d} z_k}{(2\sqrt{\pi})^d\Gamma\left(\frac{d}{2}+1\right)(\bar{m}_k^2 + z_k)^3} \tag{8.57}$$

Equating both sides leaves us with the first dimensionless flow of interest

$$\partial_t \bar{\lambda}_k = \frac{6\bar{\lambda}_k^2 z_k}{(2\sqrt{\pi})^d\Gamma\left(\frac{d}{2}+1\right)(\bar{m}_k^2 + z_k)^3} - (4-d)\bar{\lambda}_k \tag{8.58}$$

We also have

$$\partial_t m_k^2 = k\frac{d}{dk}(\bar{m}_k^2 k^2) = k\left(2k\bar{m}_k^2 + k^2\frac{d}{dk}\bar{m}_k^2\right) \tag{8.59}$$

$$\frac{-\lambda_k z_k}{(2\sqrt{\pi})^d\Gamma\left(\frac{d}{2}+1\right)}\frac{k^{2+d}}{(m_k^2+z_k k^2)^2} = \frac{-\bar{\lambda}_k k^{4-d} z_k}{(2\sqrt{\pi})^d\Gamma\left(\frac{d}{2}+1\right)}\frac{k^{2+d}}{(\bar{m}_k^2 k^2 + z_k k^2)^2} \tag{8.60}$$

$$= \frac{-\bar{\lambda}_k k^2 z_k}{(2\sqrt{\pi})^d\Gamma\left(\frac{d}{2}+1\right)(\bar{m}_k^2 + z_k)^2} \tag{8.61}$$

Equating both sides leaves us with

$$\partial_t \bar{m}_k^2 = \frac{-\bar{\lambda}_k z_k}{(2\sqrt{\pi})^d\Gamma\left(\frac{d}{2}+1\right)(\bar{m}_k^2 + z_k)^2} - 2\bar{m}_k^2 \tag{8.62}$$

9 Background Field Method

9.1 Background field method in non-gauged theories

We now must introduce a method which will allow us to compute gauge-invariant effective actions. In non-gauged theories, the background field method is identical to the "field-shifting" method. We define an analogous the generating functional of disconnected diagrams $\tilde{Z}[J]$ which is a function of the fluctuating fields ϕ plus arbitrary background fields λ which can be thought of as an alternate source.

$$\tilde{Z}[J,\lambda] = \int \mathcal{D} \exp\left[i\left\{\mathcal{S}[\phi+\lambda,\lambda] + J\cdot\phi\right\}\right] \tag{9.1}$$

The source J only couples to fluctuating fields. We can then of course define the analogous generator of connected diagrams, macroscopic averaged field, and effective action.

$$\tilde{\mathcal{W}}[J,\lambda] = -i\ln\tilde{Z}[J,\lambda], \qquad \tilde{\Phi} = \frac{\delta\tilde{\mathcal{W}}[J,\lambda]}{\delta J}, \qquad \tilde{\Gamma}[\tilde{\Phi},\lambda] = \tilde{\mathcal{W}}[J,\lambda] - J\cdot\tilde{\Phi} \tag{9.2}$$

We now shift field in Eq.(9.1) $\phi \to \phi - \lambda$. This allows us to relate the conventional and background field generating functionals. We find

$$\to \int \mathcal{D}[\phi] \exp\left[i\left\{\mathcal{S}[\phi,\lambda] + J\cdot(\phi-\lambda)\right\}\right] \tag{9.3}$$

$$= Z[J]\exp\left[-iJ\cdot\lambda\right] \tag{9.4}$$

Taking the logarithm of both sides

$$\tilde{\mathcal{W}}[J,\lambda] = \mathcal{W}[J] - J\cdot\lambda \tag{9.5}$$

Differentiating both sides and noting Eqs.(9.2),(5.8) leaves us with the relation

$$\tilde{\Phi} = \Phi - \lambda \tag{9.6}$$

Plugging into the defined background-field effective action

$$\tilde{\Gamma}[\tilde{\Phi},\lambda] = \mathcal{W}[J] - J\cdot\lambda - J\cdot(\Phi-\lambda) = \mathcal{W}[J] - J\cdot\Phi = \Gamma[\Phi] \tag{9.7}$$

But from Eq.(9.6), $\tilde{\Phi} = \Phi - \lambda$ thus,

$$\tilde{\Gamma}[\tilde{\Phi},\lambda] = \Gamma[\tilde{\Phi}+\lambda] \tag{9.8}$$

As a special case we can take $\tilde{\Phi} = 0$ such that

$$\tilde{\Gamma}[0,\lambda] = \Gamma[\lambda] \tag{9.9}$$

This indicates that the effective action can be determined by computing $\tilde{\Gamma}[0,\lambda]$. The background-field effective action is just a conventional effective action computed in the presence of the background field λ.

9.2 Background field method in gauged theories

9.3 Gauge Invariant Generating Functional

As we saw in the previous section, for a Euclidean guage theory the generating functional of disconnected Green's functions is given by

$$Z[J,\tilde{A}] = \int \mathcal{D}[a^a_\mu] \det\left[\frac{\delta G^a}{\delta \omega^b}\right] \exp\left[-\mathcal{S}[\tilde{A}_\mu + a_\mu] - i\int d^dx \left\{\frac{1}{2\xi}G^aG^a + \mathbf{J}^a_\mu \tilde{A}^a_\mu\right\}\right] \tag{9.10}$$

Where the gauge field $\mathcal{A}^a_\mu$ has been split up into sum of a background field $\tilde{A}$ and a fluctuating field a^a_μ

$$\mathcal{A}^a_\mu = \tilde{A}^a_\mu + a^a_\mu, \tag{9.11}$$

G^a is a gauge fixing term, and

$$\mathcal{S}[\mathcal{A}^a_\mu] = \mathcal{S}[\tilde{A}^a_\mu + a^a_\mu] = \frac{1}{4}\int d^dx \mathcal{F}^a_{\mu\nu}\mathcal{F}^a_{\mu\nu} \tag{9.12}$$

With

$$\begin{aligned} \mathcal{F}^a_{\mu\nu} &= \partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu + \tilde{g} f^{abc}\mathcal{A}^b_\mu \mathcal{A}^c_\nu \\ &= \tilde{F}^a_{\mu\nu} + (D_\mu[\tilde{A}])^{ab} a^b_\nu - (D_\nu[\tilde{A}])^{ab} a^b_\mu + \tilde{g} f^{abc} a^b_\mu a^c_\nu \end{aligned} \tag{9.13}$$

Where

$$\tilde{F}^a_{\mu\nu} = \partial_\mu \tilde{A}^a_\nu - \partial_\nu \tilde{A}^a_\mu + \tilde{g} f^{abc}\tilde{A}^b_\mu \tilde{A}^c_\nu \tag{9.14}$$

$$(D_\mu[\tilde{A}])^{ab} a^b_\nu = \partial_\mu a^a_\nu - i\tilde{g}\tilde{A}^b_\mu (T_b)^{ac} a^c_\nu \tag{9.15}$$

is invariant under the gauge transformation

$$\delta\mathcal{A}^a_\mu = \frac{1}{\tilde{g}}(D_\mu[\mathcal{A}])^{ab}\delta\omega^b \tag{9.16}$$

We can split this transformation such that the background field $\tilde{A}$ transforms inhomogeneously whereas the fluctuation a_μ transforms homogeneously as a tenor in the adjoint representation.

$$\delta\tilde{A}^a_\mu = \frac{1}{\tilde{g}}(D_\mu[\tilde{A}])^{ab}\delta\omega^b \tag{9.17}$$

$$\delta a^a_\mu = i\delta\omega^b (T_b)^{ac} a^c_\mu \tag{9.18}$$

We should note that T_b here are the generators in the adjoint representation ($T^\dagger_b = T_b$) which are related to the real structure constants by

$$f^{abc} = i(T_b)^{ac} = -i(T_b)^{ca} \tag{9.19}$$

$$[T^a, T^b] = if^{abc}T^c \tag{9.20}$$

The source term in Eq.(9.10) is invariant under the transformation in Eq.(9.16) provided that the source J_μ^a transforms homogeneously as an adjoint tensor. We will choose to work in the background gauge with the gauge condition

$$G^a = (D_\mu[\tilde{A}])^{ab} a_\mu^b \tag{9.21}$$

10 $\beta-$functions for SU(N) Yang-Mills Theory in d-dimensions

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