AXIONS

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1 Axions

Instantons and the θ-vacuum [27][28][29]

Due to topological conservation laws, ultimately set by the gauge boundary conditions, non-Abelian gauge fields give rise to an infinite number of degenerate vacua in real space separated by potential barriers and distinguished by their respective winding number n. Finite energy soliton solutions in 4-D Euclidean space called "instantons" provide non-zero transition amplitudes from vacuum with winding number n to vacua with winding number $m = n + \nu$ where ν represents the winding number of the instanton defined as

$$\nu \equiv \frac{g^2}{16\pi^2} \int d^4x \mathrm{Tr}[\tilde{\mathbf{F}}_{\mu\nu}\mathbf{F}^{\mu\nu}] \tag{1}$$

$$\tilde{\mathbf{F}}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathbf{F}^{\rho\sigma} \tag{2}$$

Where $\mathbf{F}_{\mu\nu}$ is the gauge field strength tensor and g is a coupling constant. According to the semi-classical WKB formalism the transmission coefficient for a particle with energy E tunneling through a potential barrier is given by

$$T(E) = \exp\Big[-\frac{1}{\hbar}\int_a^b \big[2m(V-E)\big]^{\frac{1}{2}}\Big]dx = \exp\big[-S_E\big] \tag{3}$$

Where S_E is the Euclidean action. Given the Euclidean action for the instanton solution we can calculate the transition probability from one vacuum to the next. For an instanton with winding number $\nu = 1$ we have

$$S_E^{inst} = -\frac{1}{2}\int d^4x \mathrm{Tr}[\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}] = \frac{8\pi^2}{g^2} \tag{4}$$

Thus the transition probability that arises due to the instanton solution for vacua with winding number n to $n+1$ is approximately

$$T(E) = \exp\Big[-\frac{8\pi^2}{g^2}\Big] = \exp\Big[-\frac{\pi}{\alpha_s}\Big] \tag{5}$$

In QCD this is not a small number and these field configurations must be included into the path integral. The tunneling effects from instantons gives our theory access to an infinite set of vacua. The actual vacuum of our theory will be a superposition of these infinite vacua which we can index by n. Under a gauge transformation Ω corresponding to the instanton solution with $\nu = 1$ we have

$$\Omega\,|n\rangle = |n+1\rangle \tag{6}$$

This means that in general the vacuum won't be invariant under gauge transformations. To remedy this problem we construct the θ vacuum.

$$|\theta\rangle = \sum_{n=-\infty}^{+\infty} e^{-in\theta} |n\rangle \tag{7}$$

Which under the same gauge transformation

$$\begin{aligned}\Omega|\theta\rangle &= \sum_n e^{-in\theta} |n+1\rangle \\ &= \sum_{n=-\infty}^{+\infty} e^{-i\theta\big((n+1)-1\big)} |n+1\rangle \\ &= \sum_{n'=-\infty}^{+\infty} e^{-i\theta(n'-1)} |n'\rangle \\ &= e^{i\theta} \sum_{n'=-\infty}^{+\infty} e^{-in'\theta} |n'\rangle = e^{i\theta} |\theta\rangle\end{aligned} \tag{8}$$

We see that the θ vacuum only changes by a phase under a gauge transformation and thus is invariant. We also notice that because $[\Omega, H] = 0$ the θ vacuum must also be an energy eigenstate of the Hamiltonian. Different values of the θ-vacua are orthogonal and consist of entirely different "worlds" with different energies. Hence, there is no gauge-invariant transformation which can change the value of the θ-vacuum and we are always confined to one θ-vacuum within our theory. We can express the general vacuum to vacuum transition amplitude

$$\langle \theta_{\text{out}} | \theta_{\text{in}} \rangle = \sum_{m,n} e^{i\theta(m-n)} \langle m|n\rangle = \sum_\nu e^{i\theta\nu} \left[\sum_n \langle (\nu+n)_{\text{out}} | n_{\text{in}} \rangle \right] \tag{9}$$

Which shows that the amplitude is the sum of all the possible transitions among vacua with different winding numbers (in the same θ-vacuum). Additionally, each amplitude is multiplied by a phase factor $e^{i\theta\nu}$. Using the path integral formalism we can show how this effect can be incorporated into the Lagrangian. The transition amplitude from $|\theta\rangle$ to $|\theta'\rangle$ is given by

$$\begin{aligned}\langle \theta | e^{-iH(t,t')} | \theta' \rangle &= \sum_{m,n} e^{im\theta} e^{-in\theta'} \langle m | e^{-iH(t,t')} | n \rangle \\ &= \sum_{m,n} e^{im(\theta-\theta')} e^{i\theta'(m-n)} \int (\mathcal{D}A_\mu)_{(m-n)} \exp\left[i \int d^4x (\mathcal{L} + J_\mu A^\mu) \right]\end{aligned} \tag{10}$$

We are required to stay in the same θ-vacuum

$$= \delta(\theta' - \theta) \sum_{\nu} e^{i\nu\theta} \int (\mathcal{D}A_\mu)_\nu \exp\left[i \int d^4x (\mathcal{L} + J_\mu A^\mu) \right]$$

Replacing ν with Eq.(1) we obtain

$$= \delta(\theta' - \theta) \sum_{\nu} \int (\mathcal{D}A_\mu)_\nu \exp\left[i \int d^4x \big(\mathcal{L} + \theta \frac{g^2}{16\pi^2} \mathrm{Tr}[\tilde{\mathbf{F}}_{\mu\nu} \mathbf{F}^{\mu\nu}] + J_\mu A^\mu \big) \right] \tag{11}$$

We can see explicitly that the effects resulting from the theta vacuum can be incorporated into our theory by adding an additional term to the effective Lagrangian density

$$\mathcal{L}_{\text{eff}} = \mathcal{L} + \mathcal{L}_\theta = \mathcal{L} + \theta \frac{g^2}{16\pi^2} \mathrm{Tr}[\tilde{\mathbf{F}}_{\mu\nu} \mathbf{F}^{\mu\nu}] \tag{12}$$

There are some interesting features of this new $\mathcal{L}_\theta$ term. For one, $\mathcal{L}_\theta$ is not invariant under parity and time reversal transformations. We can see this explicitly by showing how $\mathcal{L}_\theta$ transform under parity. Under a parity transformation $\mathbf{P}$ we require

$$\begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} t \\ -x_1 \\ -x_2 \\ -x_3 \end{pmatrix} \tag{13}$$

Consequently we see under the metric $(+ - - -)$, the term $\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathbf{F}^{\rho\sigma}\mathbf{F}_{\mu\nu}$ under $\mathbf{P}$ transforms as follows

$$\mathbf{P}\mathbf{F}^{\mu\nu} = \mathbf{F}_{\mu\nu} \tag{14}$$

$$\mathbf{P}\mathbf{F}_{\mu\nu} = \mathbf{F}^{\mu\nu} \tag{15}$$

Under a general transformation $\epsilon_{\mu\nu\rho\sigma}$ transforms

$$\epsilon'_{\mu\nu\rho\sigma} = A_{\mu l} A_{\nu m} A_{\rho n} A_{\sigma o} \epsilon_{lmno} = \det(\mathbf{A}) \epsilon_{\mu\nu\rho\sigma} \tag{16}$$

A parity transformation has $\det(\mathbf{P}) = -1$ thus

$$\mathbf{P}\epsilon_{\mu\nu\rho\sigma} = \det(\mathbf{P}) \epsilon^{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma} \tag{17}$$

Putting it all together we see

$$\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathbf{F}^{\rho\sigma}\mathbf{F}_{\mu\nu} \to \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\mathbf{F}_{\rho\sigma}\mathbf{F}^{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathbf{F}^{\rho\sigma}\mathbf{F}_{\mu\nu} \tag{18}$$

Thus under a parity transformation

$$\mathcal{L}_\theta \to -\mathcal{L}_\theta \tag{19}$$

and we see $\mathcal{L}_\theta$ is not invariant under parity. Using similar arguments we can see that $\mathcal{L}_\theta$ is also not invariant under time reversal. This extra term in the Lagrangian also causes the neutron to pick up an electric dipole moment proportional to θ. The detailed calculation isn't very illuminating so I'll just outline the calculation. The interactions between the neutron, proton, and pion can be expressed in the following effective interaction Lagrangian

$$L_{\pi NN} = \pi^a \bar{\Psi}_N (i\gamma^5 g_{\pi NN} + \bar{g}_{\pi NN})\tau^a \Psi_N \tag{20}$$

Where Ψ_N is the proton-neutron isospin doublet. The first term gives the normal Yukawa coupling to the psuedoscalar pions. The second term is CP violating and must be proportional to θ. Taking isospin to SU(3) and using baryon mass relations one can show that the Yukawa coupling between the pion and nucleons can be expressed as

$$\bar{g}_{\pi NN} = \frac{2m_s m_u m_d}{f_\pi(m_u+m_d)}(M_\Xi - M_N)\theta \approx 0.04\theta \tag{21}$$

Loops of pions such as in Figure (1) generate a neutron electric dipole moment. Cutting off the UV divergences at m_N gives

$$d_N = \frac{m_N}{4\pi^2} g_{\pi NN}\bar{g}_{\pi NN} \ln\frac{m_N}{m_\pi} = (5.2\times 10^{-16}\text{e cm})\theta \tag{22}$$

The current bound on the neutron electric dipole moment is $|d_N| < 2.9 \times 10^{-26}$e cm. This leaves

$$\theta < 10^{-10} \tag{23}$$

Why this CP violating parameter is so small when we have large CP violation in the weak sector is known as the strong CP problem.

2 Axial Anomaly

U(1) Problem

In the massless quark limit (and no knowledge of the chiral anamoly), the Lagrangian of QCD is seen to have a large global chiral symmetry expressed as $\text{U}(3)_L \times \text{U}(3)_R =$ $\text{SU}(3)_L \times \text{SU}(3)_R \times \text{U}(1)_V \times \text{U}(1)_A$. When the quark mass terms are included into the theory the chiral symmetry is explicitly broken down to $\mathrm{SU}(3) \times \mathrm{U}(1)_V \times \mathrm{U}(1)_A$. The eight broken generators associated with the breakdown of $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \rightarrow \mathrm{SU}(3)$ each have an associated psuedo-Nambu-Goldstone boson which are identified with the octet of psuedoscalar mesons $\pi^{\pm}, \pi^0, K^{\pm}, K^0, \bar{K}^0, \eta, \eta'$. If $\mathrm{U}(1)_A$ is an exact symmetry we would expect each hadron state to be degenerate with another hadron state (equal baryon number, spin, and strangeness) but with opposite parity. This partiy doubling is not seen in the hadron spectrum and thus we suspect $\mathrm{U}(1)_A$ is spontaneously broken in the same way that $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ is broken. In the case where $\mathrm{U}(1)_A$ is spontaneously broken we expect two isoscalar 0^- mesons. One being the η and the other with mass comparable to the pion. In 1975 Weinberg [3] showed, using current algebra, that this unidentified isoscalar psuedoscalar Nambu-Goldstone boson would be required to have a mass $m \leq \sqrt{3} m_\pi$. The η is a isoscalar 0^- however $m_\eta \gg m_\pi$ thus the η is ruled out. The expected NGB is not seen in experiment and was dubbed by Weinberg the "U(1) Problem". Because we don't observe this particle it would seem that perhaps there is no $\mathrm{U}(1)_A$ symmetry within the strong interactions.

As we saw in our previous discussion, the tunneling probability due to instanton effects are not negligible at Λ_{QCD} and we must include the $\mathcal{L}_\theta$ term in the QCD Lagrangian. In the chiral basis, the QCD Lagrangian becomes

$$\mathcal{L}_{\mathrm{QCD}} = i\bar{\psi}\not{D}\psi + M_q\bar{\psi}\psi - \frac{1}{4}G^{\mu\nu}G_{\mu\nu} + \theta\frac{g^2}{32\pi^2}\tilde{G}_{\mu\nu}G^{\mu\nu} \tag{24}$$

$$\not{D} = \gamma^\mu D_\mu = \gamma^\mu(\partial_\mu - ig\mathbf{A}_\mu) \tag{25}$$

$$G^a_{\mu\nu} \equiv \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \tag{26}$$

Where $\psi \equiv \psi_L + \psi_R$, $\psi_L \equiv \frac{1-\gamma^5}{2}\psi$, $\psi_R \equiv \frac{1+\gamma^5}{2}\psi$, $G^a_{\mu\nu}$ is the gluon field strength tensor, $\mathbf{A}_\mu \equiv A^a_\mu T^a$ where A_μ is the gluon gauge field and $T^a \equiv \lambda^a/2$ are the Gell-man matrices associated with SU(3). In the limit where the quark masses go to zero $M_q \rightarrow 0$ we see that the Lagrangian has a large chiral symmetry expressed as $\mathrm{U}(3) \times \mathrm{U}(3) = \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V \times \mathrm{U}(1)_A$ or equivalently

$$\psi \rightarrow \exp\big(i(\theta_a T^a + \gamma^5\beta_a T^a + \theta_V + \alpha\gamma^5)\big)\psi \tag{27}$$

For our purposes we are interested in in how the Lagrangian changes under an infinitesimal axial transformation identified by

$$\psi \rightarrow \psi' = e^{-i\alpha\gamma^5}\psi \approx \psi + \delta\psi + \mathcal{O}(\delta\alpha^2) = \psi - i\delta\alpha\gamma^5\psi$$

$$\psi^\dagger \to \psi'^\dagger = e^{i\alpha\gamma^5}\psi^\dagger \approx \psi^\dagger + \delta\psi^\dagger + \mathcal{O}(\delta\alpha^2) = \psi^\dagger + i\delta\alpha\gamma^5\psi^\dagger \tag{28}$$

$$A_\mu \to A'_\mu = e^{-i\alpha\gamma^5}A_\mu \approx A_\mu + \delta A_\mu + \mathcal{O}(\delta\alpha^2) = A_\mu - f^{abc}\delta\alpha\gamma^5 A_\mu \tag{29}$$

We should note that the above transformation acts differently on left-handed and right-handed quarks

$$\delta\psi_R = -i\delta\alpha\gamma^5\psi_R = \gamma^5\left(\frac{1+\gamma^5}{2}\right)\psi = i\delta\alpha\gamma^5\psi_R \tag{30}$$

$$\delta\psi_L = -i\delta\alpha\gamma^5\psi_L = \gamma^5\left(\frac{1-\gamma^5}{2}\right)\psi = -i\delta\alpha\gamma^5\psi_L \tag{31}$$

We can express the change in the Lagrangian from an infinitesimal transformation as

$$\delta\mathcal{L} = \sum_n \left[\frac{\partial\mathcal{L}}{\partial\phi_n} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\right]\delta\phi_n + \partial_\mu\underbrace{\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\delta\phi_n\right]}_{\delta\alpha J^\mu_{5,n}} \tag{32}$$

Which is summed over n-fields (both matter and gauge fields). Let's now look at the QCD Lagrangian without $\mathcal{L}_\theta$ and calculate $\delta\mathcal{L}$ under an infinitesimal chiral transformation. First we rewrite the massless QCD Lagrangian with expanded notation for calculational convenience

$$\mathcal{L} = i\bar{\psi}_i(\delta_{ij}\not{\partial}_\mu - ig\not{A}_\mu T^a_{ij})\psi_j - \frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu)^2 \tag{33}$$

When the equations of motion are satisfied, the first term in Eq (32) is equal to zero. The second term is just $\partial_\mu J^\mu_5 = \partial_\mu(J^\mu_{5,A_\mu} + J^\mu_{5,\psi})$

$$J^\mu_{5,\psi} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\frac{\delta\psi}{\delta\alpha} = \bar{\psi}\gamma^\mu\gamma^5\psi \tag{34}$$

The color currents are conserved

$$\partial_\mu J^\mu_{5,\psi} = \partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi) = 0 \tag{35}$$

For the gauge fields we have

$$J^\mu_{5,A_\mu} = \frac{\partial\mathcal{L}}{\partial(\partial_\lambda A_\sigma)}\frac{\delta A_\mu}{\delta\alpha} = \frac{\partial\mathcal{L}}{\partial F_{\mu\nu}}\frac{\partial F_{\mu\nu}}{\partial(\partial_\lambda A_\sigma)}\frac{\delta A_\mu}{\delta\alpha}$$

From Eq (29) we see that

$$\frac{\delta A_\mu}{\delta\alpha} = -f^{abc}A_\mu \tag{36}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\mu)} = -\frac{1}{2}F_{\mu\nu}\frac{\partial(\partial_\mu A^a_\nu)}{\partial(\partial_\lambda A_\sigma)}\frac{\partial(\partial_\nu A^a_\mu)}{\partial(\partial_\lambda A_\sigma)} = -\frac{1}{2}F_{\mu\nu}(\delta^\mu_\lambda\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\lambda)$$
$$= -\frac{1}{2}(F_{\lambda\sigma} - F_{\sigma\lambda}) = -F_{\mu\nu} \tag{37}$$

Where in the last equality I used the fact that $F_{\sigma\lambda} = -F_{\lambda\sigma}$ and then renamed the indices. Thus

$$J^\mu_{5,A_\mu} = f^{abc}\gamma^5 A^b_\mu F^c_{\mu\nu} \tag{38}$$

Plugging this current into the equations of motion it is easy to verify that this current is conserved.

$$\partial_\mu J^\mu_{5,A_\mu} = 0 \tag{39}$$

This would indicate that the chiral current is conserved $\partial_\mu J^\mu_5 = 0$. It turns out that this current actually contains an anomalous divergence resulting from quantum corrections. This chiral anomaly was first discovered by Jackiw and Bell [28] and then verified, generalized, and extended to higher orders in perturbation theory by Adler [2] in 1969. While investigating the decay rate of $\pi^0 \to 2\gamma$ they found that the regulator needed in order to derive consequences of the conservation of the neutral axial vector current for one loop diagrams breaks chiral symmetry. This broken regulator is the source of the anomalous divergence. If we calculate the contributions of the anomaly where the chiral current and two gluon fields couple as in Figure (2), the conserved chiral current attains a total divergence.

$$\partial_\mu J^\mu_5 = 2N_f\frac{g^2}{32\pi^2}\tilde{G}_{\mu\nu}G^{\mu\nu} \tag{40}$$

Where N_f is the number of fermions being considered. This total divergence confirms our initial hunch from the U(1) problem that $\mathrm{U}(1)_A$ in fact isn't a symmetry of the strong interactions.

We can now see that the infinitesimal change resulting from from the chiral transformation is

$$\delta L = 2\delta\alpha N_f\frac{g^2}{32\pi^2}\tilde{G}_{\mu\nu}G^{\mu\nu} \tag{41}$$

Comparing this with $\mathcal{L}_\theta$ in Eq (24) we can see that under a continuous chiral transformation the θ parameter is transformed to $\theta \to \theta + 2\alpha N_f$. This property of the chiral transformation which allows us to change the θ parameter will be important when we attempt to resolve the strong CP problem.

2.1 EW Contibutions to θ [12]

Let's take a look at another contribution to the θ parameter which results directly from the above discussion. The up and down quark Yukawa mass terms in the Standard model are of the form

$$\mathcal{L}_{\text{mass}} = -\Gamma^d_{ij}\bar{\Psi}^i_L \Phi d^j_R - \Gamma^u_{ij}\bar{\Psi}^i_L \tilde{\Phi} u^j_R + h.c. \tag{42}$$

Where i and j are generation indices, $\Gamma^{u,d}_{ij}$ are 3x3 Yukawa mass matrices, Ψ^i_L are left-handed quark doublets, and Φ is the complex Higgs doublet. When the Higgs field gets a vacuum expectation value (VEV) v and the electroweak symmetry is spontaneously broken the mass terms become

$$\mathcal{L}_{\text{mass}} = -\frac{v}{\sqrt{2}}\big[\bar{d}_L\Gamma_d d_R + \bar{u}_L\Gamma_u u_R\big] + \text{h.c.} \tag{43}$$

The Yukawa matrices in Eq.(43) are general complex matrices which aren't necessarily hermitian. It would appear that we are going to get complex mass values $m\bar{q}q$ $m \in \mathbb{C}$. Of course, the masses must be real in order to be physical. To fix this we can redefine our fields via a phase transformation $q_R \to e^{i\beta}q_R$ after we diagonalize the Yukawa matrices. Let's see this in action, to diagonalize the Yukawa matrices we can use unitary matrices $\mathrm{K}_{u,d}$ and $\mathrm{U}_{u,d}$ and rewrite the Yukawa matrices as

$$\Gamma_d \to U_d M_d K^\dagger_d, \quad \Gamma_u \to U_u M_u K^\dagger_u \tag{44}$$

Where M_u, M_d are the diagonalized mass matrices. Plugging these into Eq (43)

$$\mathcal{L}_{\text{mass}} = -\frac{\nu}{\sqrt{2}}\big[\bar{d}_L U_d M_d K^\dagger_d d_R + \bar{u}_L U_u M_u K^\dagger_u u_R\big] + h.c. \tag{45}$$

To do away with $K^\dagger$ and U we can change our basis via chiral rotations for the right-handed fields $d_{R,i} \to e^{i\beta_i}d_{R,i}$, $u_{R,i} \to e^{i\beta_i}u_{R,i}$ and non-chiral rotations for the left-handed fields, $d_L \to U_d d_L$, $u_L \to U_d d_L$. As we saw above, each time we perform a chiral rotation, the value of the θ parameter will change by $\theta \to \theta + 2\beta_i$. The total change in θ is given by

$$\theta_{EW} = \sum_i \beta_i = \arg\,\det(K_d K_u) = \arg\big[\det(M_u M_d)\det(\Gamma_d\Gamma_u)\big] = -\arg\,\det(\Gamma_d\Gamma_u) \tag{46}$$

We see that there are two contributions to the θ-parameter, one from instanton effects and the other from the mass phases in the EW interactions. These two effects are independent and have no reason to cancel. So, the total θ which will appear in L_θ is expressed by

$$\bar{\theta} \equiv \theta + \theta_{EW} \tag{47}$$

2.2 Fujikawa method

The quantum anomaly is best understood as a symmetry breaking by quantization procedure. In the path-integral formalism anomalies arise when the Lagrangian of the generating functional is invariant under a group of transformations but the path-integral measure is not.

Considering a general $SU(n)$ non-abelian gauge theory coupled to left-handed Dirac spinors.

$$\mathcal{L} = i\bar{\psi}_L \not{D} \psi_L - \frac{1}{4}\mathrm{Tr}[\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}] \tag{48}$$

$$D_\mu = \partial_\mu + ig\mathbf{A}_\mu, \qquad \mathbf{A}_\mu = A^a_\mu T^a \tag{49}$$

$$\mathbf{F}_{\mu\nu} = F^a_{\mu\nu}T^a, \qquad F^a_{\mu\nu} \equiv \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \tag{50}$$

Making a transformation to Euclidean space ($x^0 \to -ix^4, \mathbf{A}_0 \to i\mathbf{A}_4, \gamma^0 \to -i\gamma^4$, $g^{\mu\nu} \equiv \mathrm{diag}(+1,-1,-1,-1) \to \mathbf{g}^{\mu\nu} \equiv \mathrm{diag}(-1,-1,-1,-1)$)

$$\not{D} = g^{\mu\nu}\gamma_\nu D_\mu \to \mathbf{g}^{\mu\nu}\gamma_\nu D_\mu = -i\gamma^4\left(-\frac{1}{i}\partial_4 - g\mathbf{A}_4\right) + \gamma^k D_k \tag{51}$$

$$= \gamma^4(\partial_4 + ig\mathbf{A}_4) + \gamma^k D_k \equiv \gamma^4 D_4 + \gamma^k D_k \tag{52}$$

With these definitions the γ-matrices become anti-hermitian[1]

$$(\gamma^\mu)^\dagger \equiv \left((-i\gamma^4)^\dagger, \gamma^{1\dagger}, \gamma^{2\dagger}, \gamma^{3\dagger}\right) = -\gamma^\mu \tag{53}$$

And thus, the covariant derivative operator $\not{D}$ becomes hermitian, defined by the scalar product[2]

$$\langle\Phi|\not{D}\Psi\rangle = \int d^4x \Phi^\dagger(x)\not{D}\Psi(x) = \langle\not{D}\Phi|\Psi\rangle \tag{54}$$

Where Φ and Ψ are Dirac spinors. Explicitly,

$$(ig\not{\mathbf{A}}_\mu)^\dagger = -ig\not{\mathbf{A}}^\dagger_\mu = -ig\mathbf{A}^\dagger_\mu\gamma^{\mu\dagger} = ig\not{\mathbf{A}}_\mu \tag{55}$$

[1] Note that this is not the case in Minkowski space where

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$$

[2] The hermitcity of an operator must be accompanied by a corresponding scalar product to denote in what Hilbert space the operator is Hermitian.

Where I have used the fact that $(A^a_\mu T^a)^\dagger = A^{a*}_\mu T^{a\dagger} = A^a_\mu T^a$.

$$\int d^4x \Phi^\dagger \partial\!\!\!/ \Psi = -\int d^4x \Phi^\dagger \overleftarrow{\partial\!\!\!/}\, \Psi = -\int d^4x \partial\!\!\!/^\dagger \Phi^\dagger \Psi = -\int d^4x \partial^\dagger_\mu \gamma^{\mu\dagger} \Phi^\dagger \Psi \tag{56}$$

$$= \int d^4x \partial\!\!\!/ \Phi^\dagger \Psi \tag{57}$$

Where in the first equality I have integrated by parts and in the second equality I have used the operator property[3] $\langle \Phi |\, \overleftarrow{\partial\!\!\!/} = \langle \partial\!\!\!/^\dagger \Phi |$. Thus, $D\!\!\!\!/ = D\!\!\!\!/^\dagger$.

Under conjugate-transposition,

$$\partial_4^\dagger = \partial_4 \qquad \partial_k^\dagger = \partial_k \tag{58}$$

$$\mathbf{A}_4{}^\dagger = \mathbf{A}_4, \qquad \mathbf{A}_k^\dagger = \mathbf{A}_k \tag{59}$$

$$\gamma^{4\dagger} = -\gamma^4, \qquad \gamma^{k\dagger} = -\gamma^k \tag{60}$$

We can now expand the Dirac fields ψ_L and $\bar{\psi}_L$ in terms of eigenfunctions of the hermitian operator $D\!\!\!\!/$

$$\psi_L(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) = \sum_n a_n \langle x|n\rangle \tag{61}$$

$$\bar{\psi}_L(x) = \sum_{n=1}^{\infty} \bar{b}_n \phi_n^\dagger(x) = \sum_n \bar{b}_n \langle n|x\rangle \tag{62}$$

Where

$$D\!\!\!\!/ \phi_n(x) = \lambda_n \phi_n(x) \tag{63}$$

$$\int d^4x \phi_n^\dagger(x) \phi_m(x) = \delta_{nm} \tag{64}$$

In this basis our Dirac action is diagonalized.

$$\int d^4x i\bar{\psi}_L D\!\!\!\!/ \psi_L = \int d^4x \sum_{n,m}^{\infty} i\bar{b}_n \phi_n^\dagger D\!\!\!\!/ a_m \phi_m = \int d^4x \sum_{n,m}^{\infty} i\bar{b}_n a_m \phi_n^\dagger \lambda_m \phi_m \tag{65}$$

$$= \sum_{n,m}^{\infty} i\lambda_m \bar{b}_n a_m \int d^4x \phi_n^\dagger \phi_m = \sum_{n,m}^{\infty} i\lambda_m \bar{b}_n a_m \delta_{nm} = \lim_{N\to\infty} \sum_n^N i\lambda_n \bar{b}_n a_n \tag{66}$$

[3]Given a ket $|\Psi\rangle$ and an operator Λ the operator Λ acting on $|\Psi\rangle$ is given by $\Lambda|\Psi\rangle = |\Lambda\Psi\rangle$. The adjoint or corresponding bra is given by $\langle \Lambda\Psi| = \langle\Psi|\Lambda^\dagger$

The Euclidean path integral is given by

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi[\mathcal{D}A^a_\mu] \exp\left[\int d^4x \left[i\bar{\psi}_L \not{D}\psi_L - \frac{1}{4}\mathrm{Tr}[\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}]\right]\right] \tag{67}$$

The Dirac measure can be written

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} = [\det\langle n|x\rangle \det\langle x|n\rangle]^{-1} \lim_{n\to\infty}\prod_{n=1}^{N} d\bar{b}_n da_n \tag{68}$$

$$= \left[\det\int d^4x \phi_n^\dagger \phi_m\right]^{-1} \lim_{n\to\infty}\prod_{n=1}^{N} d\bar{b}_n da_n \tag{69}$$

$$= [\det\delta_{nm}]^{-1} \lim_{n\to\infty}\prod_{n=1}^{N} d\bar{b}_n da_n \tag{70}$$

$$= \lim_{n\to\infty}\prod_{n=1}^{N} d\bar{b}_n da_n \tag{71}$$

Where I have used the fact that for square matrices A and B of equal size $\det(A)\det(B) = \det(AB)$.

Under local infinitesimal chiral transformations

$$\psi_L(x) \to \psi'_L(x) = e^{i\alpha(x)\gamma^5}\psi_L(x) \approx \psi_L(x) + i\alpha(x)\gamma^5\psi_L(x) \tag{72}$$

$$\bar{\psi}_L(x) \to \bar{\psi}'_L(x) = \bar{\psi}_L(x)e^{i\alpha(x)\gamma^5} \approx \bar{\psi}_L(x) + i\alpha(x)\bar{\psi}_L(x)\gamma^5 \tag{73}$$

Where in Euclidean space γ^5 is given as

$$\gamma^5 = -\gamma^1\gamma^2\gamma^3\gamma^4 = (\gamma^5)^\dagger, \qquad (\gamma^5)^2 = \mathbb{1} \tag{74}$$

Expanding in our complete basis of eigenfunctions

$$\psi'_L(x) = \sum_n \phi_n(x)\langle n|\psi'_L\rangle = \sum_n a_n\phi_n(x) + i\alpha(x)\gamma^5\sum_n a_n\phi_n(x) \tag{75}$$

Multiplying from the left with $\phi_n^\dagger(x)$ and integrating over space

$$\int d^4x \phi_L^\dagger \psi'_L(x) \equiv a'_n = a_n + i\sum_m a_m \int d^4x \phi_n^\dagger(x)\alpha(x)\gamma^5\phi_m(x) \tag{76}$$

Likewise,

$$\bar{\psi}'_L(x) = \sum_n \phi_n^\dagger(x) \langle n|\bar{\psi}'_L\rangle = \sum_n \bar{b}_n \phi_n(x) + i \sum_n \bar{b}_n \phi_n^\dagger(x)\alpha(x)\gamma^5 \tag{77}$$

Multiplying from the right with $\phi_n(x)$ and integrating over space

$$\int d^4x \bar{\psi}(x)\phi_n(x) \equiv \bar{b}'_n = \bar{b}_n + i \sum_m \bar{b}_m \int d^4x \phi_m^\dagger(x)\alpha(x)\gamma^5\phi_n(x) \tag{78}$$

The integration measure becomes

$$\lim_{N\to\infty} \prod_{n=1}^{N} d\bar{b}'_n da'_n = \left[\det\left(\delta_{nm} + i \int d^4\phi_n^\dagger(x)\alpha(x)\gamma^5\phi_n(x)\right)\right]^{-2} \lim_{N\to\infty} \prod_{n=1}^{N} d\bar{b}_n da_n \tag{79}$$

Using the relations

$$\det\left[e^A\right] = e^{\mathrm{Tr}[A]} \tag{80}$$

or

$$\det[A] = e^{\mathrm{Tr}[\ln A]} \tag{81}$$

and

$$(\det[A])^b = \det[A^b] \tag{82}$$

The determinant in Eq.(79) becomes

$$\exp\left\{\mathrm{Tr}\left[-2\ln\left(\delta_{nm} + i\int d^4x\phi_n^\dagger(x)\alpha(x)\gamma^5\phi_m(x)\right)\right]\right\} \tag{83}$$

Expanding $\ln(\cdots)$ in a Taylor series and keeping to first order in α i.e.

$$\ln(1+\epsilon x) \approx \epsilon x + \mathcal{O}(\epsilon^2) \qquad \text{for } \epsilon \ll 1 \tag{84}$$

$$= \exp\left\{\mathrm{Tr}\left[-2i\int d^4x\phi_n^\dagger(x)\alpha(x)\gamma^5\phi_m(x)\right]\right\} \tag{85}$$

$$= \exp\left[-2i \lim_{N\to\infty}\sum_{n=1}^{N}\int d^4x\phi_n^\dagger(x)\alpha(x)\gamma^5\phi_n(x)\right] \tag{86}$$

This is non-other than the Jacobian $\mathcal{J}$ for the Dirac measure under a chiral transformation

$$\mathcal{D}\bar{\psi}'\mathcal{D}\psi' = \mathcal{J}\mathcal{D}\bar{\psi}\mathcal{D}\psi \tag{87}$$

$$\mathcal{J} \equiv \exp\left[-2i \lim_{N\to\infty} \sum_{n=1}^{N} \int d^4x \phi_n^\dagger(x)\alpha(x)\gamma^5\phi_n(x)\right] \tag{88}$$

To evaluate explicitly we need to regulate the sum over modes in a gauge invariant way. We can replace the so-called "mode cut-off" $\lim_{N\to\infty}\sum_{n=1}^{N}$ with an eigenvalue cut-off $\lim_{\Lambda\to\infty} f(\lambda_n^2/\Lambda^2)$ where Λ is a regularisation scale. While both regularizations can be made arbitrarily equivalent with a suitable choice of regulator f the latter is desirable because we can show the regulator independence of our result.

$$\begin{aligned}\mathcal{J} &= \exp\left[-2i \lim_{\Lambda\to\infty} \int d^4x\alpha(x) \sum_{n=1}^{\infty} \phi_n^\dagger(x)\gamma^5 f\left(\frac{\lambda_n^2}{\Lambda^2}\right)\phi_n(x)\right] \\ &= \exp\left[-2i \lim_{\Lambda\to\infty} \int d^4x\alpha(x) \sum_{n=1}^{\infty} \phi_n^\dagger(x)\gamma^5 f\left(\frac{\not{D}^2}{\Lambda^2}\right)\phi_n(x)\right]\end{aligned} \tag{89}$$

Performing a unitary transformation from the $\phi_n(x)$ basis to a plane-wave basis $e^{ik\cdot x}$ we can extract the gauge field dependence directly[4].

$$= \exp\left\{-2i \lim_{\Lambda\to\infty} \int d^4x\alpha(x)\mathrm{Tr}\left[\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x}\gamma^5 f\left(\frac{\not{D}^2}{\Lambda^2}\right)e^{ik\cdot x}\right]\right\} \tag{90}$$

Where the trace is over spinor and gauge group generator indices. The $\not{D}^2$ operator can be rewritten as

$$\not{D}^2 = \gamma^\mu D_\mu \gamma^\nu D_\nu = D_\mu D_\nu \gamma^\nu\gamma^\mu = D_\mu D_\nu\left(\frac{[\gamma^\mu,\gamma^\nu]}{2} + \frac{\{\gamma^\mu,\gamma^\nu\}}{2}\right) \tag{91}$$

Using the Clifford algebra relation $\{\gamma^\mu,\gamma^\nu\} = 2\mathbf{g}^{\mu\nu}$

$$= D_\mu D_\nu \mathbf{g}^{\mu\nu} + D_\mu D_\nu \frac{[\gamma^\mu,\gamma^\nu]}{2} = D^\mu D_\mu + \frac{1}{2}(D_\mu D_\nu \gamma^\mu\gamma^\nu - D_\mu D_\nu\gamma^\nu\gamma^\mu) \tag{92}$$

$$= D^\mu D_\mu + \frac{1}{2}\gamma^\mu\gamma^\nu[D_\mu, D_\nu] = D^\mu D_\mu - \frac{ig}{2}\gamma^\mu\gamma^\nu \mathbf{F}_{\mu\nu} \tag{93}$$

[4] The sum over the n modes can be considered as a trace over the operator $\gamma^5 f(\not{D}^2/\Lambda^2) \equiv \mathcal{O}$ in the $\phi_n(x)$ eigenspinor basis of fields. $\mathrm{Tr}[\mathcal{O}] = \sum_n \phi_n^\dagger(x)\mathcal{O}_x\phi_n(x) = \sum_n \langle\phi_n|x\rangle\langle x|\mathcal{O}|\phi_n\rangle = \sum_n \langle x|\mathcal{O}|\phi_n\rangle\langle\phi_n|x\rangle = \langle x|\mathcal{O}|x\rangle = \int \frac{dk}{2\pi}\langle x|k\rangle\langle k|\mathcal{O}|x\rangle = \int \frac{dk}{2\pi} e^{-ikx}\mathcal{O}_k e^{ikx}$ where $\mathcal{O}_x$ and $\mathcal{O}_k$ describe the operator $\mathcal{O}$ in the x and k basis.

Where in the second line I have renamed indicies in the last term of the first line $\mu \leftrightarrow \nu$. Because $\gamma^5 f(\not{D}^2/\Lambda)$ is hermitian we can evaluate it in position or momentum space $\langle k|\mathcal{O}|x\rangle$. If we choose to evaluate in position space we have

$$D^2 = D^\mu D_\mu = (\partial^\mu + ig\mathbf{A}^\mu)(\partial_\mu + ig\mathbf{A}_\mu) = \partial^2 + ig(\partial^\mu \mathbf{A}_\mu + \mathbf{A}^\mu \partial_\mu) - g^2\mathbf{A}^2 \tag{94}$$

If we assume the gauge fields vanish at infinity or at the boundary we can integrate by parts in the second term after adding and subtracting $\mathbf{A}^\mu\partial_\mu$

$$ig(\partial^\mu \mathbf{A}_\mu + \mathbf{A}^\mu \partial_\mu) = ig(\partial^\mu \mathbf{A}_\mu - \mathbf{A}^\mu \partial_\mu + 2\mathbf{A}^\mu\partial_\mu) = ig(2\partial^\mu \mathbf{A}_\mu + 2\mathbf{A}^\mu\partial_\mu) \tag{95}$$

Thus,

$$D^2 e^{ik\cdot x} = \left[-k^2 - 2g\mathbf{A}^\mu k_\mu - g^2\mathbf{A}^2 + 2ig\partial^\mu \mathbf{A}_\mu\right] e^{ik\cdot x} = \left[-(k_\mu + g\mathbf{A}_\mu)^2 + 2ig\partial^\mu\mathbf{A}_\mu\right] e^{ik\cdot x} \tag{96}$$

The Jacobian becomes

$$\begin{aligned}\mathcal{J} = \exp\bigg\{ &- 2i\int d^4x\alpha(x) \\ &\times \lim_{\Lambda\to\infty} \mathrm{Tr}\left[\int \frac{d^4k}{(2\pi)^4}\gamma^5 f\left(\frac{-(k_\mu + g\mathbf{A}_\mu)^2}{\Lambda^2} + \frac{2ig\partial^\mu\mathbf{A}_\mu}{\Lambda^2} - \frac{ig}{2}\frac{\gamma^\mu\gamma^\nu\mathbf{F}_{\mu\nu}}{\Lambda^2}\right)\right]\bigg\}\end{aligned} \tag{97}$$

Shifting the integration variable $k_\mu \to (k_\mu - g\mathbf{A}_\mu)$ and then performing a scale transformation $k^\mu \to k^\mu\Lambda$

$$\begin{aligned}= \exp\bigg\{ &- 2i\int d^4x\alpha(x) \\ &\times \lim_{\Lambda\to\infty} \Lambda^4\mathrm{Tr}\left[\int \frac{d^4k}{(2\pi)^4}\gamma^5 f\left(-k^2 + \frac{2ig\partial^\mu\mathbf{A}_\mu}{\Lambda^2} - \frac{ig}{2}\frac{\gamma^\mu\gamma^\nu\mathbf{F}_{\mu\nu}}{\Lambda^2}\right)\right]\bigg\}\end{aligned} \tag{98}$$

Next we expand[5] our regulator function in powers of $1/\Lambda$ about k^2 noting that any term in the expansion proportional to $1/\Lambda^5$ or higher order will vanish in the limit $\Lambda \to \infty$ and the identities

$$\mathrm{Tr}\left[\gamma^5\right] = \mathrm{Tr}\left[\gamma^5\gamma^\mu\gamma^\nu\right] = 0 \tag{99}$$

[5]
$$f(x) = f(k^2) + f'(x)\bigg|_{x=k^2}(x-k^2) + \frac{f''(x)}{2!}\bigg|_{x=k^2}(x-k^2)^2 + \cdots$$

The expansion gives one contribution to the Jacobian

$$= \exp\Bigg\{-2i\int d^4x\alpha(x)\mathrm{Tr}\left[\gamma^5\left(\frac{ig}{2}\gamma^\mu\gamma^\nu\mathbf{F}_{\mu\nu}\right)^2\int\frac{d^4k}{(2\pi)^4}\frac{f''\left(-k^2\right)}{2!}\right]\Bigg\}\Bigg|_{k^2=0} \tag{100}$$

$$= \exp\Bigg\{\frac{ig^2}{4}\int d^4x\alpha(x)\mathrm{Tr}\left[\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\mathbf{F}_{\mu\nu}\mathbf{F}_{\rho\sigma}\right]\int\frac{d^4k}{(2\pi)^4}f''\left(-k^2\right)\Bigg\}\Bigg|_{k^2=0} \tag{101}$$

We compute separately each component

$$\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\mathbf{F}_{\mu\nu}\mathbf{F}_{\rho\sigma}] = \mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma]\mathrm{Tr}[\mathbf{F}_{\mu\nu}\mathbf{F}_{\rho\sigma}] \tag{102}$$

Where the first trace is over spinorial components and the second trace is over gauge generator indices. To show

$$\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = -4\epsilon^{\mu\nu\rho\sigma} \tag{103}$$

We note that the only contributing case is when $\mu \neq \nu \neq \rho \neq \sigma$. In this case, the trace is proportional to the antisymmetric symbol due to the anticommutation of gamma matrices with differing indices. To find the proportionality constant we can simply do a test case of $\mu\nu\rho\sigma = 1234$.

$$\mathrm{Tr}[\gamma^5\gamma^1\gamma^2\gamma^3\gamma^4] = -\mathrm{Tr}[\gamma^1\gamma^2\gamma^3\gamma^4\gamma^1\gamma^2\gamma^3\gamma^4] \tag{104}$$

$$= \mathrm{Tr}[\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0] \tag{105}$$

$$= (-1)^3\mathrm{Tr}[\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3] \tag{106}$$

$$= -(-1)^2\mathrm{Tr}[\gamma^1\gamma^1\gamma^2\gamma^3\gamma^2\gamma^3] \tag{107}$$

$$= (-1)\mathrm{Tr}[\gamma^2\gamma^2\gamma^3\gamma^3] \tag{108}$$

$$= -\mathrm{Tr}[\mathbb{1}_{4\times 4}] = -4 \tag{109}$$

Thus, the proportionality constant is -4 and

$$\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = -4\varepsilon^{\mu\nu\rho\sigma} \tag{110}$$

Next,

$$\int\frac{d^4k}{(2\pi)^4}f''\left(-k^2\right) = \frac{2\pi^2}{\Gamma(2)(2\pi)^4}\int_0^\infty dkk^3f''(-k^2) = \frac{1}{16\pi^2}\int_0^\infty dxxf''(x) \tag{111}$$

$$= \frac{1}{16\pi^2}\left[xf'(x)\Big|_0^\infty - \int_0^\infty f'(x)\right] \tag{112}$$

$$= \frac{1}{16\pi^2} \tag{113}$$

The last equality provides the conditions of validity for our regulator

$$f(0) = 1, \quad f(\infty) = 0, \quad xf'(x)\Big|_{x=0} = xf'(x)\Big|_{x=\infty} = 0 \tag{114}$$

Any regulator with these properties is a valid. Plugging everything in, the Jacobian of the chiral transformation becomes

$$\mathcal{J} = \exp\left\{ -\int d^4x\alpha(x)\frac{ig^2}{16\pi^2}\mathrm{Tr}\left[\epsilon^{\mu\nu\rho\sigma}\mathbf{F}_{\mu\nu}\mathbf{F}_{\rho\sigma}\right]\right\} \tag{115}$$

Choosing the normalization

$$\mathrm{Tr}[T^aT^b] = \frac{1}{2}\delta^{ab} \tag{116}$$

We have

$$\mathcal{J} = \exp\left\{ -\int d^4x\alpha(x)\frac{ig^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}\right\} \tag{117}$$

$$= \mathcal{J} = \exp\left\{ -\int d^4x\alpha(x)\frac{ig^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}\right\} \tag{118}$$

We could also just as well perform the calculation in momentum space. The covariant derivative becomes $\partial_\mu \to i\hat{p}$, and $\hat{p}\,|k\rangle = k\,|k\rangle$

$$D_\mu = (\partial_\mu + ig\mathbf{A}_\mu) \rightsquigarrow (i\hat{p}_\mu + ig\mathbf{A}_\mu) \tag{119}$$

3 Axion to aisle θ for strong CP cleanup [26][16][13][19]

As we saw above, the strong interactions seem to conserve CP and experimental measurement of the neutron electric dipole moment place strong bounds on the value of the θ parameter. Perhaps the most elegant solution to the strong CP problem was provided by Roberto Peccei and Helen Quin in 1977. There main idea was to mimic the properties of the chiral transformations effects on θ to somehow eliminate the CP violating parameter all together. They showed that this could be achieved given that at least one of the quarks obtains its mass via a Yukawa coupling to a scalar field. Naturally, we can identify this scalar field with a Higgs field in some electroweak theory. The theory acquires a new chiral U(1) symmetry termed U(1)$_{PQ}$ which is spontaneously broken at some energy scale f_a when the scalar field associated with the PQ symmetry acquires a VEV . The original Peccei-Quinn theory involves an extension of the Standard Model with the inclusion of an additional Higgs doublet. In order to avoid problems with FCNCs we say that one Higgs doublet couples exclusively to charge 2/3 up quarks and the other couples to only charge -1/3 down quarks. Although the original axion from the standard axion model has been ruled out by experiment, it is informative to look at this model to understand the physics and then extend the same arguments to invisible axion models. We begin with the Lagrangian $\mathcal{L}_{QCD} + \mathcal{L}_{WS} + \mathcal{L}_H$ where the first two terms are the standard QCD and Weinberg-Salam models and $\mathcal{L}_H$ contains the couplings involving the extra set of Higgs fields.

$$\begin{aligned}\mathcal{L}_H = (D_\mu\Phi_1)^\dagger(D_\mu\Phi_1) + (D_\mu\Phi_2)^\dagger(D_\mu\Phi_2) - V(\Phi_1,\Phi_2)\\ - \Gamma^1_{ij}(\bar{d}_{Ri}\tilde{\Phi}_1\Psi_{Lj} + \bar{\Psi}_{Li}\Phi_1 d_{Rj}) - \Gamma^2_{ij}(\bar{u}_{Ri}\tilde{\Phi}_2\Psi_{Lj} + \bar{\Psi}_{Li}\Phi_1 u_{Rj})\end{aligned} \tag{120}$$

$$\Psi_L = \begin{bmatrix} u_L \\ d_L \end{bmatrix}, \Phi_1 = \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix}, \Phi_2 = \begin{bmatrix} \phi'^0 \\ \phi'^- \end{bmatrix} \tag{121}$$

$$D_\mu = \partial_\mu - igW^a_\mu\tau^a - \frac{1}{2}ig'B_\mu \tag{122}$$

Where $\Phi_{1,2}$ are the two complex Higg's multiplets, $\tilde{\Phi}_{1,2} = i\sigma_2\Phi^*_{1,2}$, $\Gamma^{1,2}_{ij}$ are the associated 3×3 Yukawa matrices, B_μ is the hypercharge gauge boson, and W^a_μ are the SU(2) gauge bosons. If the scalar fields carry the Peccei-Quinn charge Q_{PQ} then we can transform them via the $\mathrm{U(1)_{PQ}}$ transformation.

$$\Psi_L \to e^{-i\alpha_L}\Psi_L, \quad u_R \to e^{-i\alpha_{u_R}}, \quad d_R \to e^{-i\alpha_{d_R}}d_R \tag{123}$$

$$\Phi_k \to e^{-i\alpha_k}\Phi_k \quad \text{for k=1,2} \tag{124}$$

In order for the Lagrangian to be invariant under $\mathrm{U(1)_{PQ}}$ the phases must satisfy

$$\alpha_1 = \alpha_L - \alpha_{d_R}, \quad \alpha_2 = \alpha_L - \alpha_{u_R} \tag{125}$$

We see that the Yukawa interaction cannot be $\mathrm{U(1)_{PQ}}$ in the SM because we need at least two Higgs doublets to satisfy Eq (125). We are interested in the properties of the axion, so we will focus on the Nambu-Goldstone boson sector of the theory. We can choose the Higgs potential $V(\Phi_1,\Phi_2)$ so that both Φ_1 and Φ_2 obtain vacuum expectation values and as a result produce four Nambu-Goldstone bosons in the absence of couplings to the gauge fields. Three of these particles are absorbed in the Higgs mechanism and give mass to the $W^\pm_\mu$, and Z_μ. The fourth Nambu-Goldstone boson is identified as the "axion" or "Higglet" [6]. We can conveniently write the axion

[6] In the literature the axion is sometimes called the higglet as proposed by Bjorken.

field independently from the other components as a common phase field in Φ_1 and Φ_2 which is orthogonal to the weak hypercharge.

$$\Phi_1 = \frac{\nu_1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \exp\left(\frac{ia}{x\nu}\right), \quad \Phi_2 = \frac{\nu_2}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \exp\left(\frac{ixa}{\nu}\right) \tag{126}$$

Where $x = \frac{\nu_1}{\nu_2}$ and $\nu = \sqrt{\nu_1^2 + \nu_2^2}$. After symmetry breaking, the axion Lagrangian becomes

$$\begin{aligned} \mathcal{L}_a = m_W^2 W_\mu^+ W_\mu^- + \frac{1}{2} m_z^2 Z_\mu^2 + \frac{1}{2}(\partial_\mu a)^2 - m_{d_j} \left(\bar{d}_R \text{exp}\left(\frac{-ia}{x\nu}\right) d_L + \bar{d}_L \text{exp}\left(\frac{ia}{x\nu}\right) d_R \right) \\ - m_{u_j} \left(\bar{u}_R^j \text{exp}\left(\frac{-ixa}{\nu}\right) u_L^j + \bar{u}_L^j \text{exp}\left(\frac{ixa}{\nu}\right) u_R^j \right) \end{aligned} \tag{127}$$

The last two terms can be rewritten

$$\mathcal{L}_a \supset -m_{d_j} \bar{d} \text{exp}\left(\frac{ia\gamma^5}{x\nu}\right) d - m_{u_j} \bar{u}^j \text{exp}\left(\frac{ixa\gamma^5}{\nu}\right) u^j \tag{128}$$

$$= -m_{ui} \left[\bar{u}_i u_i \text{cos}\left(\frac{xa}{\nu}\right) + \bar{u} i \gamma^5 u_i \text{sin}\left(\frac{xa}{\nu}\right) \right] - m_{di} \left[\bar{d}_i d_i \text{cos}\left(\frac{a}{x\nu}\right) + \bar{d} i \gamma^5 d_i \text{sin}\left(\frac{a}{x\nu}\right) \right] \tag{129}$$

Where the mass matrices has been diagonalized. The original U$(1)_{PQ}$ symmetry in Eq (120) can still be realized after symmetry breaking through transformations where the axion field translates

$$a \rightarrow a + \alpha f_a \tag{130}$$

Where f_a is the axion order parameter associated with the breakdown of U$(1)_{\text{PQ}}$. In the standard axion model f_a is set to coincide with the electroweak breakdown $\nu \approx 250$ GeV. The assignment of PQ (chiral) charge for each field is model-dependent, following the original paper [4], they took all left handed fermions to have Q_{PQ}=0 and d_R, u_R are assumed to only couple to Φ_1 and Φ_2, respectively. In this case we see from Eqs (125) and (127) that the axion Lagrangian is invariant under U$(1)_{\text{PQ}}$ if

$$\alpha_{d_R} = \frac{\alpha}{x}, \quad \alpha_{u_R} = \alpha x \tag{131}$$

Now that we know how the fields change under U$(1)_{PQ}$ we can calculate the PQ current.

$$J^\mu_{\text{PQ}} = \nu \partial_\mu a + \frac{1}{x} \bar{d}_R \gamma^\mu d_R + x \bar{u}_R \gamma^\mu u_R \tag{132}$$

The current is conserved via the classical equations of motions but as we know from the chiral transformations studied in the previous section, this current is afflicted by the same total divergence due to the chiral anomaly and proportional to the number of fermion doublets N

$$\partial_\mu J^\mu_{\text{PQ}} = N_f \left(x + \frac{1}{x}\right) \frac{g^2}{32\pi^2} \tilde{G}_{\mu\nu} G^{\mu\nu} \tag{133}$$

Now we need to relate the axion back to $\mathcal{L}_{\bar{\theta}}$. We can do this easily by removing the axion from Φ_i via a chiral transformation on the right-handed quarks

$$u_R \to \exp\left(\frac{-ixa}{\nu}\right) u_R, \quad d_R \to \exp\left(\frac{-ia}{x\nu}\right) d_R \tag{134}$$

You probably recognize at this point that this chiral transformation is the same one we performed in our earlier discussion where we were diagonalizing the Yukawa mass matrices via chiral phase rotations. We see that this chiral transformation equivalently changes $\bar{\theta}$

$$\bar{\theta} \to \bar{\theta} + \left(x + \frac{1}{x}\right) \frac{a}{\nu} \equiv \bar{\theta} + N_m \frac{a}{\nu} \tag{135}$$

Where N_m is a model-dependent coefficient. When it's all said and done, we add an additional term to the axion Lagrangian to ensure both the chiral anomaly in Eq (133) and the transformation property in Eq (135) are satisfied. The resulting term creates an effective potential for the axion which is periodic in the effective vacuum angle.

$$V_{\text{eff,a}} = \left(\theta + \frac{a}{\nu} N\right) \frac{g^2}{32\pi^2} \tilde{G}_{\mu\nu} G^{\mu\nu} \tag{136}$$

We are interested where this potential is minimized to get the axion VEV

$$\langle \frac{\partial V_{\text{eff,a}}}{\partial a} \rangle = \frac{N}{\nu} \frac{g^2}{32\pi^2} \langle \tilde{G}_{\mu\nu} G^{\mu\nu} \rangle \Big|_{\langle a \rangle} = 0 \tag{137}$$

Peccei and Quinn showed that the axion VEV must take the form

$$\langle a \rangle = -\bar{\theta} \frac{\nu}{N} \tag{138}$$

This can be more easily understood by first considering what the axion VEV could be before including the effects of the color anomoly. Because the axion is a phase field the $\text{U}(1)_{\text{PQ}}$ symmetry allows

$$0 \leq N \frac{\langle a \rangle}{\nu} \leq 2\pi \tag{139}$$

When we include the effects of the color anomaly it generates a potential for the axion which is periodic in the effective vacuum angle $\bar{\theta} + \langle a \rangle N/\nu$:

$$V_{\text{eff}} \approx \cos\left(\bar{\theta} + N\frac{\langle a \rangle}{\nu}\right) \tag{140}$$

Which is minimized when

$$\langle a \rangle = -\bar{\theta}\frac{\nu}{N} \tag{141}$$

The physical axion will result from excitations around the minimum of this potential. Thus, we define

$$a = \langle a \rangle + a_{\text{phys}} \tag{142}$$

When the Lagrangian is rewritten in terms of a_{phys} the Lagrangian no longer has the CP-violating $\bar{\theta}$-term. In conclusion we see that if $\bar{\theta}$ is not initially zero, the axion field will shift to dynamically cancel the non-zero $\bar{\theta}$ and therby conserve CP, solving the strong CP problem.

3.1 Invisible Axion Models

The original axion proposed by Weinberg and Wilczek [6] (motivated by the work of Peccei and Quinn [4][5]), while a useful pedagogical tool, has been essentially ruled out by experiment (Table 1) as well as astrophysical arguments.

Reaction	Theory	Experiment
$\text{BR}(K^+ \to \pi^+ + a)$	$\approx 3 \times 10^{-5}(x + \frac{1}{x})^2$	$< 3.8 \times 10^{-8}$
$\text{BR}(J/\Psi \to a + \gamma)$	$3.7 \pm 0.8 \times 10^{-5} x^2$	$< 1.4 \times 10^{-5}$
$\text{BR}(\upsilon \to a + \gamma)$	$2.0 \pm 0.7 \times 10^{-4}(1/x)^2$	3×10^{-4}
$\tau(a \to 2\gamma)$	$\approx 0.2(100\text{keV}/m_a)^5$ sec	not detected
$\tau(a \to e^+e^-)$	$\approx 4 \times 10^{-9}(1\text{MeV}/m_a)(1 - 4m_e^2/m_a^2)^{1/2}x^3$ sec	not detected

Table 1: This table shows the experimental results which rule out the Peccei-Quinn-Weinberg-Wilczek (PQWW) axion.

Variant axion models, known as invisible axion models, where $f_a \gg \nu$ are still viable. Invisible axion models contain very light, very weakly interacting axions, hence their name. In essence, all invisible axion models are constructed using the same two ingredients

- A new complex scalar field σ is introduced into the theory which carries Q_{PQ} and has a very large expectation value
- The scalar field is a singlet under SU(2)×U(1)$_Y$ and only couples to ordinary matter through U(1)$_{\mathrm{PQ}}$ symmetry

What varies from model to model is the assignment of Q_{PQ} to ordinary matter. Two popular models within the literature, namely the KSVZ and DFSZ models, are discussed in the following.

3.1.1 KSVZ Model

In the KSVZ invisible axion model, written down by Kim [7]; Shifman, Vainshtein, and Zakharov [8], regular fermions carry no Q_{PQ} and the axion is introduced as a phase field in the scalar field σ. The theory also assumes the existence of a new (heavy) quark Q with mass $\mathrm{M}_Q \approx f_a$. The new quark is a singlet under SU(2)×U(1), carries Q_{PQ}, and couples with the σ field. The relevant Yukawa term for the new quark is

$$\mathcal{L}_{\mathrm{Yuk}}^{\mathrm{KSVZ}} = h\bar{Q}_L \sigma Q_R + h.c. \tag{143}$$

Where h is chosen to be a positive Yukawa coupling and

$$\sigma = \frac{f_a}{\sqrt{2}} \mathrm{exp}\left(\frac{ia}{f_a}\right) \tag{144}$$

As we did in the standard axion model, if we assume only Q_R carries PQ charge then by similar arguments the axion can couple to ordinary matter through the chiral anomaly

$$\mathcal{L}_a^{\mathrm{KSVZ}} = \frac{a}{f_a}\left(\frac{g^2}{32\pi^2}\tilde{G}_{\mu\nu}G^{\mu\nu} + 2e_Q^2\frac{\alpha}{4\pi}\tilde{F}_{\mu\nu}F^{\mu\nu}\right) \tag{145}$$

Where e_Q is the electric charge of the Q quark and $F_{\mu\nu}$ is the electromagnetic field strength. By construction this axion doesn't couple with leptons at tree level and only interacts with light quarks through the strong and electromagnetic anomalies. Because of these properties, this axion is sometimes referred to as the "hadronic" axion.

3.1.2 DFSZ Model

The DFSZ model, written down by Dine, Fischler, and Srednicki [9] as well as Zhitnitsky [10], is identical to the Peccei-Quinn model but with the additional complex scalar field σ. The σ field can only directly couple to ordinary fermions through the Higgs potential and couples to the Higgs fields through a quartic coupling

$$\mathcal{L}_{\sigma H} = \kappa \Phi_1^{+} C (\sigma^{\dagger})^2 \Phi_2 + h.c. \tag{146}$$

The model demands that the potential $V(\Phi_1, \Phi_2, \sigma)$ give a large VEV to σ, which in return sets $f_a \gg \nu$ and in this limit the three Higgs fields become

$$\sigma = \frac{f_a}{\sqrt{2}} \exp\left(\frac{ia}{f_a}\right) \tag{147}$$

$$\Phi_1 = \frac{\nu_1}{\sqrt{2}} \exp\left(\frac{iX_1 a}{f_a}\right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \Phi_2 = \frac{\nu_2}{\sqrt{2}} \exp\left(\frac{iX_2 a}{f_a}\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{148}$$

$$X_1 = 2\left(\frac{\nu_2}{\nu}\right)^2, \quad X_2 = 2\left(\frac{\nu_1}{\nu}\right)^2 \tag{149}$$

We can see that the quartic coupling in Eq.(146) is invariant under axion translations $a \to \alpha f_a$ as required. The DFSZ axion has a smaller interaction strength than the standard axion and interacts with leptons at tree level.

The two models discussed above differ in their coupling strengths to matter and distinguish how the potential axion behaves in the physical world. In the next section we discuss some of the consequences of these different couplings.

3.2 Axion Mass and Couplings

The axion mass was first calculated by Bardeen and Tye [11] using low energy current algebra methods. The small mass ultimately stems from total divergence in the axial current Eq (133) due to the color anomaly. Thus, the axion only attains a mass at Λ_{QCD}. Bardeen and Tye showed that the axion mass is given by

$$m_a^2 = N \frac{F_\pi^2 m_\pi^2}{f_a^2} \frac{z}{(1+z)^2} = 6.3\ \mu\text{eV}\left(N \frac{10^{12}\ \text{GeV}}{f_a}\right) \tag{150}$$

$$F_\pi = 93\ \text{MeV}, \quad z = \frac{m_u}{m_d} = 0.56 \pm 0.042\ [14] \tag{151}$$

Again, N is a model dependent coefficient. As an example the standard axion $N = N_g(x + \frac{1}{x})$ and thus $m_a \sim 75(x + 1/x)$ keV. Using effective Lagrangian techniques $m_a \sim 25$ keV [15].

Phenomenologically speaking, the most important and distinguishing feature of each invisible axion model is its coupling with the photon. We can write down an effective Lagrangian for the interaction of axions and photons

$$\mathcal{L}_{\text{eff},a\gamma} = -\frac{1}{2}(\partial_\mu a)^2 - \frac{1}{2}m_a a^2 + \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) - \frac{1}{4}g_{a\gamma\gamma} a \tilde{F}_{\mu\nu}F^{\mu\nu} \tag{152}$$

The final term on the RHS is the axion-photon interaction term and is of particular interest

$$\mathcal{L}_{a\gamma\gamma} = -\frac{1}{4}g_{a\gamma\gamma} a \tilde{F}_{\mu\nu}F^{\mu\nu} = -g_{a\gamma\gamma} a \mathbf{E}\cdot\mathbf{B} \tag{153}$$

$$g_{a\gamma\gamma} = \frac{\alpha}{2\pi f_a}\left(\frac{E}{N} - \frac{2}{3}\frac{4+z}{1+z}\right) = \frac{\alpha}{2\pi}\left(\frac{E}{N} - \frac{2}{3}\frac{4+z}{1+z}\right)\frac{\sqrt{z}}{1+z}\frac{m_a}{m_\pi F_\pi} \tag{154}$$

$$N = \sum_j Q_{\text{PQ,j}}, \quad E = 2\sum_j Q_{\text{PQ},j} e_{Q,j}^2 N_{\text{c},j} \tag{155}$$

N and E are model dependent coefficients which appear in the color and electromagentic anomaly. Q_{PQ} is the PQ charge for the jth fermion, e_Q is the electric charge on the jth fermion (in units of proton charge e), and N_c is the color degrees of freedom (3 for quarks, 1 for leptons) on the jth fermion. For the DFSV model $\frac{E}{N} = \frac{8}{3}$ and for the KSVZ model $\frac{E}{N} = 0$ (when $e_Q = 0$) [13] [16].

The axion interactions with fermions can be written most generally as

$$\mathcal{L}_{af} = -i g_{aff} \bar{\Psi}_j \gamma^5 \Psi_j a \quad \text{or} \quad \frac{C_j}{f_a}\bar{\Psi}_j \gamma^\mu \gamma^5 \Psi_j \partial_\mu a \tag{156}$$

$$g_{aff} = \frac{C_f m_f}{f_a} \tag{157}$$

Where m_f is the fermion mass and C_f is a model dependent parameter. We can also define an "axion fine structure constant" which is often used in the literature

$$\alpha_{a,j} = \frac{g_{aff}^2}{4\pi} \tag{158}$$

Table (2) contains the relevant parameter values for the DFSV and KSVZ invisble axion models which are used in most phenomenological analyses.

Parameter	DFSZ	KSVZ
$g_\gamma = g_{a\gamma\gamma}(\frac{\pi f_a}{\alpha})$	0.37	-0.96
C_e	$\frac{1}{3}\cos^2\beta$	0
C_p	$-0.10 - 0.45\cos^2\beta$	-0.39
C_n	$-0.18 + 0.39\cos^2\beta$	+0.04

Table 2: $\tan\beta = \nu_2/\nu_1$, $g_\gamma = (E/N - 1.92)/2$. This table shows the relevant model-dependent parameters. To see how the DFSZ parameters depend on $\cos^2\beta$ see Figure (3)

Its also good to note that g_γ and C_f are dimensionless while $g_{a\gamma\gamma}$ has dimensions of m^{-1}

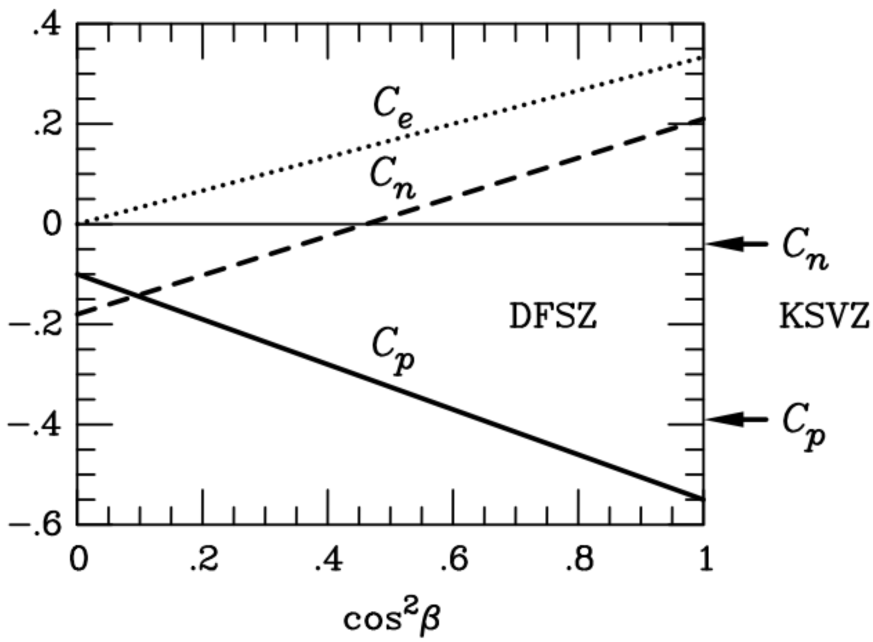

Figure 1: [16] This figure shows how the DFSZ model parameters scale with $\cos^2\beta$

4 Astrophysical and Cosmological Limits [16]

The well-motivated axion, if realized in nature, would play a prominent role in the astrophysical arena. With our knowledge of astrophysical phenomena we can place stringent bounds on axion dynamics. In some parameter spaces, the axion may be a viable dark matter candidate. I will now discuss some notable bounds based on astrophysical observation and theory.

4.0.1 Stellar Cooling and New Particle Channels

Stellar evolution is tightly correlated to the allowed nuclear and thermal processes occurring within the stellar interior. Any addition of allowed energy channels (energy sinks) within the stellar interior would have effects on the propagation of energy through the stellar interior and ultimately effecting the cooling and evolution of the star. On degenerate stars such as white dwarfs the existence of these new energy channels would cause a higher energy-loss rate and accelerate the cooling of the star. For non-degenerate stars like the sun, the impact is less obvious. The extra energy channel leads to contraction and heating until temperatures rise enough to increase nuclear burning which offsets the extra energy loss. This indicates that the extra energy channel would increase the consumption of nuclear fuel and decrease the period in which the star remains in this fuel-burning regimen. For example invisible axions may contribute to energy loss via Primakoff process in the hydrogen burning main sequence of a solar mass star. This extra energy loss from the new axion channel would cause the main sequence star to burn its hydrogen more quickly and the time spent in the main sequence would decrease. To be exact, we can produce a simple quantitative model which describes the effect of incorporating a new energy loss channel on the stellar structure. We begin with the differential equation derived via energy conservation considerations for a stellar model

$$\frac{dL_r}{dr} = 4\pi r^2 \epsilon \rho \tag{159}$$

$$\epsilon = \epsilon_{\text{nuc}} + \epsilon_{\text{grav}} - \epsilon_\nu - \epsilon_x \tag{160}$$

Where L_r is the net flux of energy through a spherical shell of radius r, and ϵ is the effective rate of local energy production with ϵ_{nuc} as the rate of nuclear energy production, ϵ_ν as the rate of energy loss by neutrino production, and ϵ_x as the energy loss due to novel particles such as invisible axions or nonstandard neutrinos with magnetic dipole moments. If we assume the perturbed configuration is given via a homology transformation in that the distance between two points is proportional to the change in the radius of the configuration i.e.

$$R' = yR \tag{161}$$

Where y is the scaling factor. Thus every point in the star is then mapped by the same transformation

$$r' = yr \tag{162}$$

The mass remains the same $M'(r') = M(r)$ and thus we can realize the transformations for density and pressure.

$$\rho'(r') = \frac{\rho(r)}{y^3}, \quad p'(r') = \frac{p(r)}{y^4} \tag{163}$$

The equation of state for a low mass, non-degenerate star is approximately given by the ideal gas law where p $\propto \frac{\rho T}{\mu}$ where μ is the average molecular weight of the electrons and nuclei within the star. Because we assumed that $\mu'(r') = \mu(r)$ the temperature and temperature gradient are

$$T'(r') = \frac{T(r)}{y}, \quad \frac{dT'(r')}{dr'} = \frac{1}{y^2}\frac{dT(r)}{dr} \tag{164}$$

most importantly we can find the transformation law for the local energy flux

$$L'(r') = \frac{L(r)}{\sqrt{y}} \tag{165}$$

Our initial assumption that the star reacts to a new particle emission through a homologous contraction places restrictions on how the local energy production can scale with density and pressure, namely

$$\epsilon \propto \rho^n T^\nu \tag{166}$$

For ϵ_{nuc}, the density has to scale with $n = 1$, and in the temperature range of the pp-chain $\nu = 4-6$. Assuming ϵ_x follows this same proportionality and if we ignore the energy loss from neutrinos (ϵ_ν) as well as the local energy gains from the contraction of the star (ϵ_{grav}) we can rewrite the rate of local energy production in Eq.(160) as

$$\epsilon = (1 - \delta_x)\epsilon_{\text{nuc}} \tag{167}$$

Where $\delta_x < 1$ and depends on the interaction strength of the new particle. From Eqs.(159), (164), and (163) we can rewrite Eq.(165)

$$L'(r') = (1 - \delta_x)\frac{L(r)}{y^{3+\nu}} \tag{168}$$

Thus,

$$y = (1 - \delta_x)^{2/(2\nu+5)} \tag{169}$$

Friedman et al. showed that in the limit $\delta_x \ll 1$ the fractional changes in stellar radius, luminosity, and interior temperature are given by

$$\frac{\delta R}{R} = \frac{-2\delta_x}{2\nu+5}, \quad \frac{\delta L}{L} = \frac{\delta_x}{2\nu+5}, \quad \frac{\delta T}{T} = \frac{2\delta_x}{2\nu+5} \tag{170}$$

We see that in this model, when a new particle is emitted the star shrinks in size, increases its surface luminosity and increases its temperature. The overall changes in stellar structure are moderate and the main feature is an increased consumption of nuclear fuel. This leads to a decreased fuel burning phase of

$$\frac{\delta\tau}{\tau} \approx -\delta_x \tag{171}$$

In general the new particle emission doesn't have the same temperature and density dependence as the nuclear burning rate which would result non-homologous transformation after the perturbation. This model still remains valid to lowest order if we interpret δ_x as an average over the entire star

$$\delta_x = \frac{L_x}{(L_x + L_\gamma)} \tag{172}$$

Where L_γ is the average photon luminosity and L_x is the averaged luminosity of the newly emitted particle which can be computed from an unperturbed solar model. The realization of the link between energy-loss and stellar evolution is a very powerful tool in placing limits on weakly-interacting particles and their couplings. By incorporating axion decay channels into solar models we can place bounds on the Yukawa couplings bewteen invisible axions with electrons, nucleons, and photons. Potential axion channels which could act as energy sinks within stellar interiors are

1. Compton scattering: $\gamma + e \to e + a$
2. Axion Bremsstrahlung: $e + Z \to e + Z + a$
3. Primakoff effect: $\gamma + Z \to Z + a$

In the following I will quote some results from analyses similar to the one above but first we need to have some idea of how stars evolve.

4.0.2 Stellar Evolution

To identify observables of stellar structure and evolution which can be used to discover and constrain new energy channels we need to understand how stars evolve through time. For our purposes I will only give a very brief and bulleted account of stellar evolution.

1. Large self-gravitating masses of hydrogen, helium, and metals contract and heat until the gas reaches a temperature where nuclear fusion begins.

2. Main Sequence (MS): Star burns helium as fuel and its brightness is completely determined by its mass.

3. Red Giant (RG): The star depletes it hydrogen reservoirs and forms a helium core with a hydrogen burning shell around it. The material around the core begins to expand and the core contracts and heats. The star goes through a number of "helium flashes" which act as the stars transition from burning hydrogen to helium as its fuel source.

4. Horizontal Branch (HB): Both the hydrogen burning shell and helium core are burning fuel as a result, the surface shrinks considerably and the surface is much bluer (hotter).

5. Asymptotic Giants/White Dwarfs (AG/WD): Helium is exhausted and a carbon-oxygen (CO) core forms with a helium shell. In high mass stars the CO begins burning as fuel and the star grows hotter and expands producing an asymptotic giant. For low mass stars, the CO core never ignites and the star releases stored heat from its interior.

6. Type I and Type II Supernova (SNI/SNII): White dwarfs in binary systems can acquire mass from the other member causing the CO core to become so hot and dense that the elements ignite leading to a (subsonic) deflagaration or (supersonic) detonation front. This sweeps through the entire WD and creates a type I supernova. Large mass stars progress until they acquire an iron core. When this core reaches its Chandrasekhar limit the core becomes unstable (due to the release of neutrinos) and collapses. A shock wave is formed at the edge of the core and propagates outward turning the implosion into a spectacular explosion termed a type II supernova.

7. Black Holes: Of course, ultra high-mass stars can collapse to form a black hole.

Evolutionary steps 2-5 can be seen distinctly on a color-magnitude diagram seen in Figure (4).

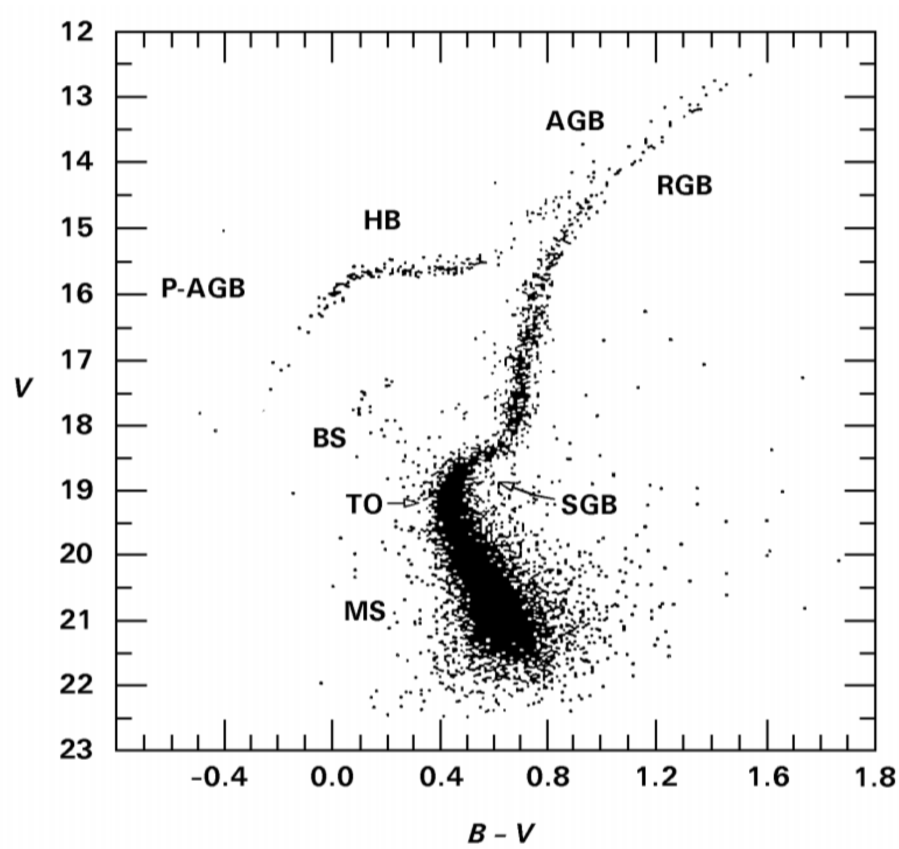

Figure 2: [16] Color-magnitude diagram for globular cluster M3 based on photometric data of 10,637 stars. MS: Main-sequence, BS: blue-stragglers, TO: Main sequence turn off, SGB: Sub-giant branch, RGB: Red-giant branch, HB: Horizontal branch, AGB: Asymptotic giant branch, and P-AGB: Post-asymptotic giant branch.

4.0.3 Horizontal branch stars in globular clusters

In the horizontal branch of the color magnitude diagram (Figure 4) the helium burning within the core and the hydrogen burning shell around the helium core are approximately balanced. This causes the total luminosity to remain constant while the surface temperature decreases. Because the core temperature of these star is $\sim 10^8 K$, axions with a mass of $m_a \leq 10$keV would contribute to the cooling process. If we consider the standard helium burning luminosity via triple-α reaction $L_{3\alpha}$ and the luminosity L_a of the non-standard energy-loss via axion Primakoff process integrated over the core then from Eq (172) we see that the period of helium burning

$t_{\rm He}$ will be reduced by an approximate factor $L_{3\alpha}/(L_a + L_{3\alpha})$. If we demand a reduction of less than 10% it translates into the requirement that $L_a \leq 0.1 L_{3\alpha}$. $L_{3\alpha} = 20 L_\odot$ is a standard value for a typical HB star. Taking the core mass to be around $0.5 M_\odot$ the core-averaged energy generation rate is $\langle \epsilon_{3\alpha} \rangle \approx 80$ erg g^{-1}s^{-1}. The energy-loss rate via axions by Primakoff process is found to be proportional to $g_{10} T^7/\rho$ where $g_{10} = g_{a\gamma\gamma}/10^{-10}$ GeV^{-1}. For a typical HB star with $\rho_4 \equiv \rho/10^4$g cm^{-3} and $T_8 \equiv T/10^8 K$, $\langle T_8^7/\rho_4 \rangle \approx 0.3$ which yields $\epsilon_a \approx g_{10}^2$ 30 erg g^{-1}s^{-1}. For $g_{10} = 1$ the helium-burning lifetime should be reduced by a factor of 80/(80+30)= 0.7. Numerical evolution sequences produced from Raffelt and Dearborn [17] with stars of mass $1.3 M_\odot$, intial Helium abundance of 25% and metalicity Z=0.02 found that the helium-burning lifetime without the axion channel was $1.2{\times}10^8$ yrs and with was modified to 0.7×10^8 yrs with $g_{10} = 1$. Axion losses on this HB model reduced the helium burning lifetime by a factor of 0.6, which is in good agreement of our analytic prediction. Thus, we have our first limit on the axion-photon coupling

$$g_{a\gamma\gamma} < 1 \times 10^{-10} \text{ GeV}^{-1} \tag{173}$$

4.0.4 Helium Ignition and White dwarfs/Red Giants

If the axion couples directly to the electron, as in the DFSZ model, the dominant emission processes would be via Compton scattering and axion Bremstughling

1. $\gamma + e \to e^- + a$
2. $e^- + Z \to Z + e^- + a$

These processes would act as energy sinks within white dwarfs and red giants causing the helium ignition to be delayed. If helium ignition is delayed, this will give more time for the core to grow and pushes the RGB to brighter stars. Thus, the RGB can act as a sensitive probe for axion emmision by looking at the brightness of the brightest stars within the branch. The constraints obtained from RGB and white dwarf numerical models on the axion-electron coupling is

$$g_{aee} < 2.5 \times 10^{-13} \tag{174}$$

From table two we find

$$f_a > 1.3 \times 10^9 \text{ GeV} \quad \text{and} \quad m_a < 4.5 \text{ meV for } \cos^2\beta = \frac{1}{2} \tag{175}$$

4.0.5 SN 1987A

The most restrictive limits on axion-nucleon couplings arise from the neutrino signal of SN 1987A. The contributing process is axion Bremsstrahlung by nucleons $N + N \to$

$N + N + a$. With a small coupling strength, the axion, if present, would amplify the role of the neutrino and reduce the burst time and gives a lower limit of the coupling strength. If the coupling strength is large, a larger number of axions are captured in the core of the neutron star and reduce the cooling effect. This produces a bottleneck on the allowed couplings. The obtained constraints are

$$3 \times 10^{-10} < g_{aNN} < 3 \times 10^{-7} \tag{176}$$

If the coupling strength becomes even larger than this, the axion as well as the neutrino would be observed. We can place further limits based on observation

$$10^{-6} < g_{aNN} < 3 \times 10^{-3} \tag{177}$$

This translates to limits on f_a and m_a

$$f_a \geq 4 \times 10^{8}\text{GeV} \quad \text{and} \quad m_a \leq 16 \text{ meV} \tag{178}$$

For $C_p = -0.4$ and $C_n = 0$. The astrophysical and cosmological bounds are summarized in Figure (5).

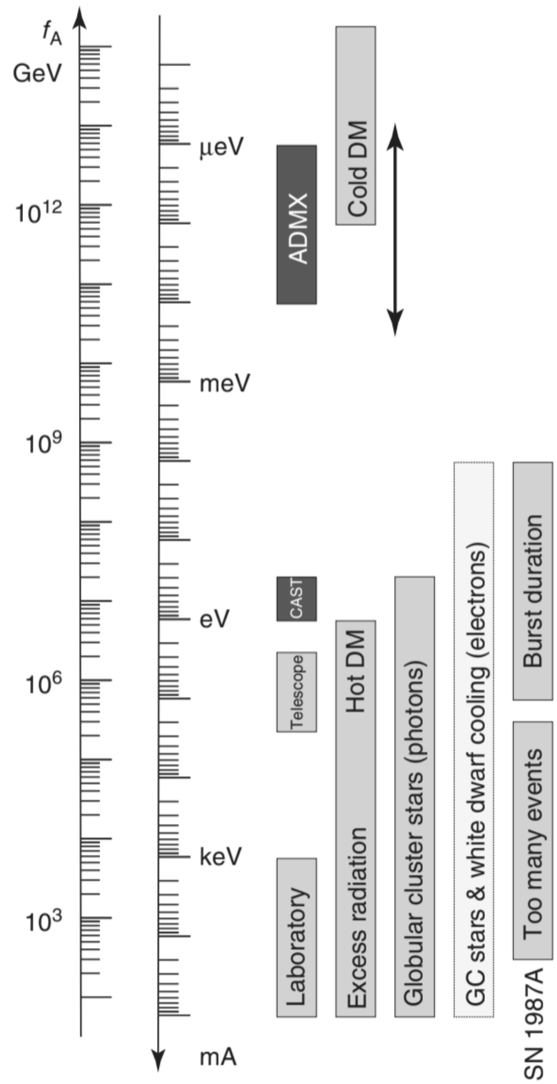

Figure 3: [13] Diagram outlining some of astrophysical and cosmological bounds on f_a and m_a.

5 Terrestrial Search for Axions

There are a number of ongoing experiments to detect the axion at a number of different mass ranges. I will describe a few experiments which are attempting to detect the axion by utilizing a reverse Primakoff reaction ($a + \gamma_{\text{virtual}} \to \gamma$). These experiments are well suited for detecting axions in a mass range of $< 10^{-5}$ eV. I focus on these because if the axion is detected in this mass range, axions could potentially be the dominant component of dark matter.

5.1 Solar Axions

The sun is potentially a large source of axions. The axion flux coming from the sun is estimated to be

$$\frac{d\Phi_a}{dE} = g_{10}^2\ 6.0 \times 10^{10}\text{cm}^{-2}\text{s}^{-1}\text{keV}^{-1}E^{2.481}e^{\frac{-E}{1.205}} \tag{179}$$

The integrated flux over the sun is

$$\Phi_a = g_{10}^2\ 3.75 \times 10^{11}\text{cm}^{-2}\text{s}^{-1} \tag{180}$$

Which means the solar axion luminosity is given by

$$L_a = g_{10}^2\ 1.85 \times 10^{-3} L_\odot, \quad L_\odot = 3.86 \times 10^{33}\ \text{erg s}^{-1} \tag{181}$$

The distribution can be seen in Figure (6). The maximum is at 3.0 keV with and average of 4.2 keV with a slight dependence on the solar model used to integrate the flux.

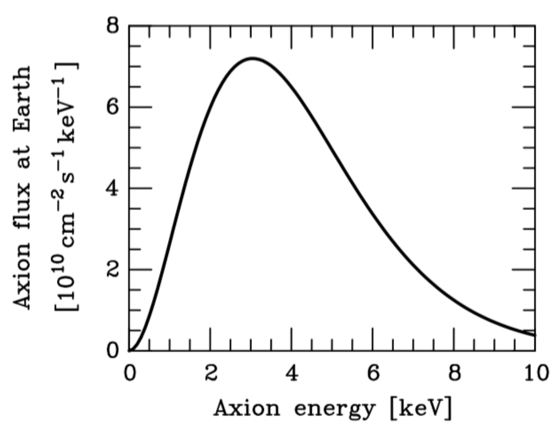

Figure 4: [16] The estimated energy distribution of axions coming from the sun.

The most sensitive axion experiments, at the present time, in the mass range of $10^{-5} \leq m_a \leq 1$ eV are axion helioscopes, i.e. magnetic solar telescopes. The underlying physical principle was first proposed by P. Sikivie in 1983. If axions are produced in the sun, they would reach the earth in $\sim$500 seconds as an approximately parallel axion beam. The helioscope on earth converts the axion via the reverse Primakoff reaction $a + \gamma_{\text{virtual}} \to \gamma$ with the virtual photon provided by an external magnetic field (Figure 7). The photons leaving the reverse reaction have the same energy and momentum as the axion and having an energy distribution as seen in Figure (6). An x-ray detector placed at the end of the magnetic field collects photons and looks for axion signatures above background.

The earliest helioscope searches were preferomed at Brookhaven and Tokyo. The first one that reached the "axion line" was carried out by CAST (CERN Axion Solar Telescope) was built in 2002 at CERN [23]. They made a detector with B=9.0 T and length 9.26m. The detector could be adjusted $\pm 8^o$ vertically and could observe for 1.5 hours during sunrise and sunset. They increased the sensitivity by filling the chamber with pressurized helium. The most recent data acquired from CAST did not see a signal but did obtain a upper limit [22].

$$g_{a\gamma\gamma} < 0.66 \times 10^{-10}\ \text{GeV}^{-1}, \quad 95\%\text{CL} \quad m_a < 0.02\ \text{eV} \tag{182}$$

5.2 Microwave Cavity Searches

Microwave cavity detectors are similar to helioscopes in that they also use the reverse Primakoff reaction to detect a axion-converted photon. The main difference is that theses detectors have a sensitivity region below $m_a \approx 10^{-5}$ eV and thus search for dark-matter axions. The conversion of non-relativistic axions produces photons in the microwave range. The idea is to place a microwave cavity in a strong magnetic field and wait for cavity modes to be excited by the axion field. Microwave cavity searches such as ADMX (Axion Dark Matter eXperiment) us a cylindrical copper tube placed within a magnetic bore where dark-matter axions can resonantly convert into real microwave photons with energy $E \approx m_a^2 + (1/2)m_a^2\beta^2$. In this way we can view the electromagnetic modes of the cavity and the free axion field modes as coupled oscillators via Eq.(153) where $\mathbf{B}$ is the external static field and $\mathbf{E}$ is from an electromagnetic cavity mode. Power is transferred from the axion field to the cavity excitations via oscillator beats caused by the weak coupling between the two fields. The ADMX experiment excluded KSVZ dark-matter axions with mass between 1.9-3.53 μeV [24]. The upgraded version of ADMX experiment will eventually cover a mass range of 1-100 μeV.

5.3 Laser Induced

Complementary to the solar and dark-matter axion searches, laser induced axion searches provide another way to obtain limits on axion parameters. Laser induced axion experiments utilize, yet again, the Primakoff reaction $\gamma + \gamma_{\text{vitual}} \rightarrow a$. The experiments try and produce axions by shining polarized laser beams through a transverse magnetic field. So called "light shining through a wall" experiments block the laser at some point on the path which only allows weakly-interacting particles to pass through (see Figure 8) [21].

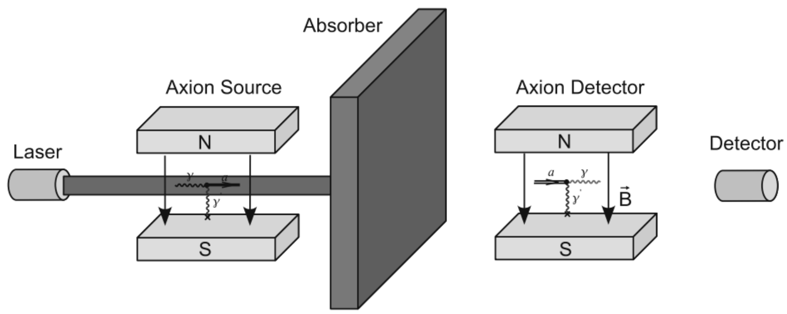

Figure 5: [19] This figure shows the experimental set up for the light shining through walls experiment.

The pioneering experiment of this kind was done by Brookhaven-Fermilab-Rutherford-Trieste (BFRT) collaboration [20]. They used two superconducting dipole magnets to provide a magnetic field strength of B=3.7 T with an optical laser of λ= 514nm and an average power of $\langle P \rangle = 3$W. The experiment found no photon regeneration but obtained a upper limit on $g_{a\gamma\gamma}$ [25].

$$g_{a\gamma\gamma} < 3.5 \times 10^{-8}\ \text{GeV}^{-1}(95\%\text{CL}), \quad m_a < 2 \times 10^{-4}\ \text{eV} \tag{183}$$

6 Axions as Dark Matter [13]

The story of the axion as a dark matter candidate begins at the cosmic temperature $\text{T} \sim \nu_{\text{PQ}}$ when the higgs field σ obtains a VEV due to the spontaneous breakdown of $\text{U}(1)_{\text{PQ}}$. The Nambu-Goldstone axion is produced in the process and the universe continues to cool. At $\text{T} \leq \Lambda_{\text{QCD}} \sim 200$ MeV the axion begins to feel the color anomaly and the winebottle potential is tilted. The axion falls down the valley of the potential and oscillates about the minimum. The anomaly producing the potential has an infinite number of valleys which the axion could fall into each with differing winding numbers. We don't know which of the n valleys the axion chooses but for simplicity we can consider the $n = 0$ valley. The axion obtains a mass at Λ_{QCD} which grows with time and reaches an asymptotic value which still remains to this day. We can describe the oscillations of the axion field by setting $\Theta = \langle a \rangle / f_a$ and approximating the Lagrangian as

$$\mathcal{L} \approx f_a^2 \left[\frac{1}{2} \dot{\Theta}^2 - \frac{1}{2} m_a^2 \Theta^2 \right] \tag{184}$$

The equation of motion produced from the above Lagrangian is that of a harmonic oscillator. However, during the time of these oscillations the universe is expanding. We can incorporate this expansion effect by rewriting our action in terms of the Robertson-Walker scale factor a(t) to $\int \mathcal{L} a(t)^3 d^4x$. The resultant equation of motion is given by

$$\ddot{\Theta} + 3H(t)\dot{\Theta} + m_a^2 \Theta = 0 \tag{185}$$

$$H(t) \equiv \frac{\dot{a}}{a} \tag{186}$$

Where a is the cosmic scale factor and H(t) is the Hubble expansion rate. When $T \gg \Lambda_{\text{QCD}}$ the axion is massless and the only solution to Eq (185) is Θ =constant. During this time, the axion is static, ultra-cold, and in a Bose condensate. When the axion feels the color anomaly and m_a begins to grow until reaching $m_a(t) \approx H(t)$. At this point, Θ begins to oscillate as a damped oscillator ($\ddot{\Theta} + 3H(t)\dot{\Theta} + H(t)\Theta = 0$). If m_a grows slowly we can approximate it by replacing $\dot{\Theta}^2$ with its average over the period $\langle \dot{\Theta} \rangle^2 \equiv \lambda(t)$, $\lambda(t) = (\dot{\Theta}^2 + m_a^2 \Theta^2)/2$. Multiplying Eq(185) by $\dot{\Theta}$ and writing in terms of $\lambda(t)$

$$\frac{\dot{\lambda}}{\lambda} + 3H(t) - \frac{\dot{m}_a}{m_a} = 0 \tag{187}$$

Solving the differential equation for $\lambda(t)$ we find

$$\lambda(t) = A \frac{m_a(t)}{a^3} \tag{188}$$

The energy density of the axion field after the effects of the color anomaly are felt is then

$$\rho_a(t) \sim f_a^2 \lambda(t) \tag{189}$$

Because the number of axions produced is invariant with expanding volume we can approximate the energy density of axions at the present time t_0

$$\rho_a(t_0) = \frac{C}{2} f_a \frac{m_\pi F_\pi T_0^3}{M_{pl} \Lambda_{\text{QCD}}} \sim \frac{f_a \Lambda_{\text{QCD}}}{M_{Pl}} T_0^3 \tag{190}$$

Where M_{Pl} is the Planck mass. The energy density cannot exceed the present cosmic density so Eq (190) is constrained

$$\rho_a < \rho_c = \frac{3H^2(t_0)}{8\pi G_{\text{Newton}}} = 11\ h^2\ \text{keV/cm}^3,\ (h \approxeq 0.72) \tag{191}$$

Thus places bounds on f_a and m_a

$$f_a < 10^{12}\ \text{GeV} \quad (\text{or } m_a \geq 10^{-5}\text{eV}) \tag{192}$$

This bound would indicate that f_a may be somewhere near the GUT scale. A detailed analysis by Sikivie gives

$$\Omega_a h^2 \approx 0.7\left(\frac{f_a}{10^{12}\ \text{GeV}}\right)^{\frac{7}{6}}\left(\frac{\langle\Theta(0)\rangle}{\pi}\right)^2 \approx 0.3\left(\frac{f_a}{10^{12}\ \text{GeV}}\right) \tag{193}$$

Where $\Omega_a = \rho_a/\rho_c$, and $\langle\Theta(0)\rangle$ is the average initial value for small oscillations about the zero point of the value. The most recent observed value of $\Omega_{DM}h^2 = 0.1187 \pm 0.0017$ by Planck, we can calculate the mass of the axion which would contribute to dark matter.

$$m_a \approxeq 6\ \mu\text{eV}\left(\frac{10^{12}\ \text{GeV}}{f_a}\right), \quad (f_a < 0.5 \times 10^{12}\ \text{GeV}) \tag{194}$$

7 Conclusion

I've reviewed many aspects of the axion in this paper. I'll end by reiterating some of the main properties of the axion.

1. If the axion exists, it solves the strong CP problem.
2. The axion is a psuedoscalar particle with $J^P = 0^-$.
3. The fundamental properties of the axion are almost essentially determined once the VEV f_a is determined.
4. The axion mass is given approximately by $m_a \approx \frac{F_\pi m_\pi}{f_a}$.
5. The axion couples to the electromagnetic field via $g_{a\gamma\gamma} a \mathbf{E} \cdot \mathbf{B}$ with strength $g_{a\gamma\gamma} \sim \alpha/(\pi f_a)$.
6. For $m_a = 10^{-3} \sim 10^{-6}$ eV, the axion could be a dominant component of dark matter.

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